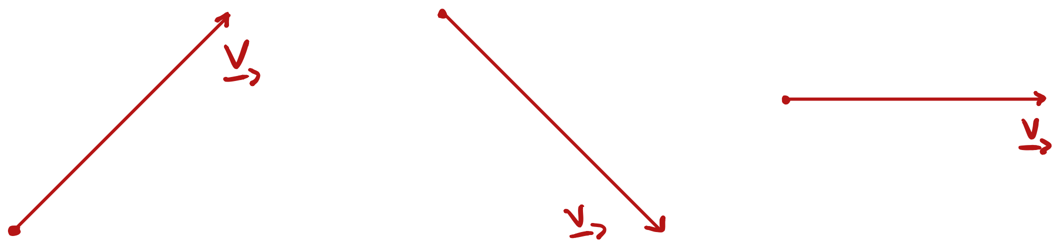
# Vectors and Reference Frames

## Vectors

- Entities that have module, direction, and sense.



- All vectors can be represented in a matrix form given a basis.

**Basis:** in an n-dimensional space, a basis is any set of n linearly independent vectors. Any vector in this space can be written by a linear combination of the elements in the basis.

Basis: $\vec{u}_1, \vec{u}_2, \vec{u}_3, \dots, \vec{u}_n$

$$\forall \vec{x} \in \mathbb{R}^n \Rightarrow \vec{x} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \alpha_3 \vec{u}_3 + \dots + \alpha_n \vec{u}_n \quad \text{for } \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$$

which can be written as:

$$\vec{x} = \underbrace{\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \end{bmatrix}}_{\text{Any basis of } \mathbb{R}^n} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\text{Representation of the vector } \vec{x} \text{ on the selected basis}}$$

Hence, it is clear that the representation of a vector depends on the selected basis.

To avoid a cumbersome notation, it is usual to omit the matrix that contains the basis, and indicate it in the vector symbol. If, in the previous example, the basis:

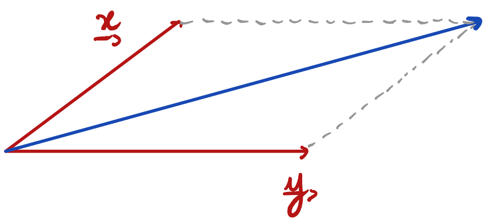
$$\vec{u}_1, \vec{u}_2, \vec{u}_3, \dots, \vec{u}_n$$

is called $a$, then we could write:

$$\underline{x}_a = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{bmatrix}$$

This means that the vector $\underset{\rightarrow}{x}$ can be represented by the coefficients $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ using the basis $a$.

It is necessary, however, to take extra caution when performing operations with vectors using this notation, because all vector **must** be represented in the same basis so that the operation makes sense. For example, let's consider the sum of two vectors:




$\underset{\rightarrow}{x} + \underset{\rightarrow}{y}$

↳ This is a new vector created using $\underset{\rightarrow}{x}$ and $\underset{\rightarrow}{y}$, and does not depend on their representation.

In this case:

Let the basis "a" be defined by $\underline{U} \triangleq [\underset{\rightarrow}{u}_1 \ \underset{\rightarrow}{u}_2 \ \underset{\rightarrow}{u}_3 \cdots \underset{\rightarrow}{u}_n]$, then

$$\underset{\rightarrow}{x} = \underline{U} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{bmatrix} \quad \text{and} \quad \underset{\rightarrow}{y} = \underline{U} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_n \end{bmatrix}$$

Hence:

$$\underset{\rightarrow}{x} + \underset{\rightarrow}{y} = \underline{U} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{bmatrix}}_{x_a} + \underline{U} \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_n \end{bmatrix}}_{y_a} = \underline{U} \underbrace{\begin{bmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \\ \alpha_3 + \beta_3 \\ \vdots \\ \alpha_n + \beta_n \end{bmatrix}}_{x_a + y_a}$$

Thus, considering the representation of the vectors in the basis $a$, we have:

$$\underline{x + y}_a = \underline{x}_a + \underline{y}_a = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_n \end{bmatrix} = \begin{bmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \\ \alpha_3 + \beta_3 \\ \vdots \\ \alpha_n + \beta_n \end{bmatrix}$$

On the other hand, let

$$\underline{V} \triangleq [\, \underline{v}_{>1} \;\; \underline{v}_{>2} \;\; \underline{v}_{>3} \; \cdots \; \underline{v}_{>n} \,]$$

be another basis called $b$ such that $\underline{y}_b = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \\ \gamma_n \end{bmatrix}$. In this case $\underline{x}_a + \underline{y}_b \neq \begin{bmatrix} \alpha_1 + \gamma_1 \\ \alpha_2 + \gamma_2 \\ \alpha_3 + \gamma_3 \\ \vdots \\ \alpha_n + \gamma_n \end{bmatrix}$

It becomes clear when writting the operation in complete matrix form, i.e. with the basis:

$$\underline{x}_{>} + \underline{y}_{>} = \underbrace{\underline{U} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{bmatrix} + \underline{V} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \\ \gamma_n \end{bmatrix}}$$

This operation is correct! You just can't simplify it any further.

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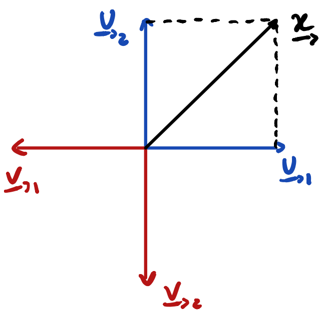
## Example: Representation of vectors in different basis




Basis $a$: $[\, \underline{u}_{>1} \;\; \underline{u}_{>2} \,]$

Basis $b$: $[\, \underline{v}_{>1} \;\; \underline{v}_{>2} \,]$

Notice that: $\underline{x}_{>} = 1 .\, \underline{u}_{>1} + 1 .\, \underline{u}_{>2} = [\, \underline{u}_{>1} \;\; \underline{u}_{>2} \,] \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{x}_{>} = (-1)\, \underline{v}_{>1} + (-1)\, \underline{v}_{>2} = [\, \underline{v}_{>1} \;\; \underline{v}_{>2} \,] \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

Finally: $\underline{x}_a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\underline{x}_b = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

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## Notation

We will use the following notation as in Hughes (2004):

$\underline{x}_{>}$ : Vector
$\underline{x}_a$ : Representation of vector $\underline{x}_{>}$ using the base $\underline{a}$.
$a$ : Scalar
$\underline{A}$ : Matrix.

# Changing basis

We verified that the vector representation depends on the selected basis of the vector space. Hence, operations between vectors can only be performed if all operands are represented on the same basis. Thus, we must develop a method to convert a vector representation between basis.

Let:

$$[\vec{e}_1 \ \vec{e}_2 \cdots \vec{e}_n] \text{ and } [\bar{\vec{e}}_1 \ \bar{\vec{e}}_2 \cdots \bar{\vec{e}}_n]$$

be two different basis of a vector space in $\mathbb{R}^n$. Given any vector $\vec{x} \in \mathbb{R}^n$, we have:

$$\vec{x} = [\vec{e}_1 \ \vec{e}_2 \cdots \vec{e}_n]\,\underline{\beta} = [\bar{\vec{e}}_1 \ \bar{\vec{e}}_2 \cdots \bar{\vec{e}}_n]\,\bar{\underline{\beta}}, \tag{1}$$

where $\underline{\beta}$ and $\bar{\underline{\beta}}$ are the representation of $\vec{x}$ in each basis. We want to find a way to obtain $\bar{\underline{\beta}}$ given $\underline{\beta}$ and the vectors in both basis.

Let $[p_{1i} \ p_{2i} \cdots p_{ni}]$ be the representation of the vector $\vec{e}_i$ in the basis $[\bar{\vec{e}}_1 \ \bar{\vec{e}}_2 \cdots \bar{\vec{e}}_n]$, then:

$$\vec{e}_i = \underbrace{[\bar{\vec{e}}_1 \ \bar{\vec{e}}_2 \cdots \bar{\vec{e}}_n]}_{\underline{E}} \underbrace{\begin{bmatrix} p_{1i} \\ p_{2i} \\ \vdots \\ p_{ni} \end{bmatrix}}_{\underline{p}_i} \triangleq \underline{E}\,\underline{p}_i \, , \quad i = 1, 2, \ldots, n$$

Hence, we can write:

$$[\vec{e}_1 \ \vec{e}_2 \cdots \vec{e}_n] = [\underline{E}\,\underline{p}_1 \ \ \underline{E}\,\underline{p}_2 \cdots \underline{E}\,\underline{p}_n] = \underline{E}\,[\underline{p}_1 \ \underline{p}_2 \cdots \underline{p}_n]$$

Thus:

$$\begin{aligned}
[\vec{e}_1 \ \vec{e}_2 \cdots \vec{e}_n] &= \underline{E}\,[\underline{p}_1 \ \underline{p}_2 \cdots \underline{p}_n] \\
&= [\bar{\vec{e}}_1 \ \bar{\vec{e}}_2 \cdots \bar{\vec{e}}_n] \cdot \underbrace{\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}}_{\underline{P}} \\
&\triangleq [\bar{\vec{e}}_1 \ \bar{\vec{e}}_2 \cdots \bar{\vec{e}}_n]\,\underline{P}
\end{aligned}$$

Using this result in (1), one gets:

Representation of $\vec{x}$ in the basis $[\bar{\vec{e}}_1 \ \bar{\vec{e}}_2 \cdots \bar{\vec{e}}_n]$

$$\vec{x} = [\vec{e}_1 \ \vec{e}_2 \cdots \vec{e}_n]\,\underline{\beta} = [\bar{\vec{e}}_1 \ \bar{\vec{e}}_2 \cdots \bar{\vec{e}}_n]\,\boxed{\underline{P}\,\underline{\beta}} = [\bar{\vec{e}}_1 \ \bar{\vec{e}}_2 \cdots \bar{\vec{e}}_n]\,\boxed{\bar{\underline{\beta}}}$$

It is known that the representation of a vector in a basis is **unique**. Hence, we can conclude that:

$$\bar{\underline{\beta}} = \underline{P}\,\underline{\beta}$$

Thus, the conversion of a representation in the basis $[\,\underline{e}_{>1}\ \underline{e}_{>2}\ \cdots\ \underline{e}_{>n}\,]$ to the representation in the basis $[\,\bar{\underline{e}}_{>1}\ \bar{\underline{e}}_{>2}\ \cdots\ \bar{\underline{e}}_{>n}\,]$ can be done by computing the following matrix:

$$\underline{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix},$$

where the $i$-th column is the representation of the vector $\underline{e}_{>i}$ in the basis $[\,\bar{\underline{e}}_{>1}\ \bar{\underline{e}}_{>2}\ \cdots\ \bar{\underline{e}}_{>n}\,]$.

↳ Each column is the representation of a vector in the source basis using the destination basis.

Analogously:

$$\underline{\beta} = \underline{Q}\,\bar{\underline{\beta}},$$

where

$$\underline{Q} = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix}$$

in which $\begin{bmatrix} q_{1i} \\ q_{2i} \\ \vdots \\ q_{ni} \end{bmatrix}$ is the representation of the vector $\bar{\underline{e}}_{>i}$ in the basis $[\,\underline{e}_{>1}\ \underline{e}_{>2}\ \cdots\ \underline{e}_{>n}\,]$.

Hence, we have:

$$\bar{\underline{\beta}} = \underline{P}\,\underline{\beta},$$
$$\underline{\beta} = \underline{Q}\,\bar{\underline{\beta}},$$

which leads to

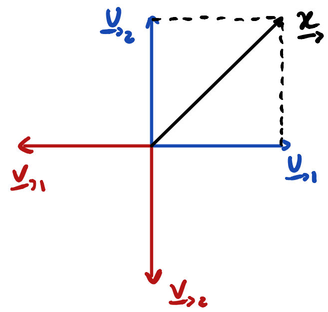
$$\bar{\underline{\beta}} = \underline{P}\,\underline{Q}\,\bar{\underline{\beta}} \quad \forall\, \underline{\beta} \in \mathbb{R}^n$$

Thus, since this is valid for all $\bar{\underline{\beta}} \in \mathbb{R}^n$, we have:

$$\underline{P}\,\underline{Q} = \boxed{\underline{I}_n} \quad \text{or} \quad \underline{P} = \underline{Q}^{-1}$$

↳ Notation: $n \times n$ identity matrix.

## Example: Changing basis




Using the same scenario in the previous example, we want to find the matrix that converts a representation in the basis a to the representation in the basis b. Hence, we have to represent the vector $\underline{u}_{>1}$ and $\underline{u}_{>2}$ using the basis composed of $\underline{v}_{>1}$ and $\underline{v}_{>2}$.

We have:

$$\underline{u}_{>1} = -\underline{v}_{>1} = (-1)\,\underline{v}_{>1} + 0\,\underline{v}_{>2} = \begin{bmatrix} \underline{v}_{>1} & \underline{v}_{>2} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\underline{u}_{>2} = -\underline{v}_{>2} = 0\,\underline{v}_{>1} + (-1)\,\underline{v}_{>2} = \begin{bmatrix} \underline{v}_{>1} & \underline{v}_{>2} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Hence:

$$D_{ba} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

↳ Notation: Matrix that converts a representation in basis a to base b.

Notice that:

$$D_{ba}\, \underline{x}_a = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \underline{x}_b$$

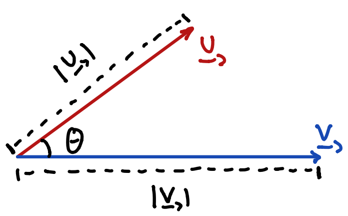
# Vectors in 3 dimensions

All the theory so far is valid for all n-dimensional vector space. However, we will focus from now on our case of interest: 3 dimensional vector space over the field of real numbers $\mathbb{R}$.

In this case, we can define some operations. In the following, let $\underline{v} = [v_1 \; v_2 \; v_3]^T$ and $\underline{u} = [u_1 \; u_2 \; u_3]^T$ be, respectively, the representation of $\underline{v}_{>}$ and $\underline{u}_{>}$ in the orthornormal basis $[\underline{e}_{>1} \; \underline{e}_{>2} \; \underline{e}_{>3}]$.

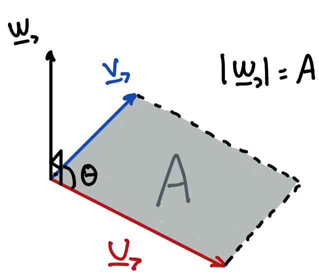
### 1) Dot product (produto escalar)

$$a = \underline{u}_{>} \cdot \underline{v}_{>} \triangleq |\underline{u}_{>}| \cdot |\underline{v}_{>}| \cdot \cos\theta$$




### 2) Cross product (produto vetorial)

$$\underline{w}_{>} = \underline{u}_{>} \times \underline{v}_{>}$$
$$|\underline{w}_{>}| = |\underline{u}_{>} \times \underline{v}_{>}| \triangleq |\underline{u}_{>}| \cdot |\underline{v}_{>}| \cdot \operatorname{sen}\theta$$




in which $\underline{w}_{>}$ is perpendicular to $\underline{u}_{>}$ and $\underline{v}_{>}$.

## 3) Vectors represented in an orthonormal basis

If the vectors are represented in an orthornormal basis, then we can write:

$$a = \underrightarrow{u} \cdot \underrightarrow{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$\underrightarrow{w} = \underrightarrow{u} \times \underrightarrow{v} = (u_2 v_3 - u_3 v_2)\,\underrightarrow{e}_1 + (u_3 v_1 - u_1 v_3)\,\underrightarrow{e}_2 + (u_1 v_2 - u_2 v_1)\,\underrightarrow{e}_3$$

$$u = |\underrightarrow{u}| = \sqrt{\underrightarrow{u} \cdot \underrightarrow{u}} = \sqrt{u_1 u_1 + u_2 u_2 + u_3 u_3}$$

## 4) Representation of vector in matrix form

Let

$$\underrightarrow{x} = x_1 \underrightarrow{e}_1 + x_2 \underrightarrow{e}_2 + x_3 \underrightarrow{e}_3$$

then

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Column matrix with the representation of the vector $\underrightarrow{x}$ in the basis $[\underrightarrow{e}_1 \;\; \underrightarrow{e}_2 \;\; \underrightarrow{e}_3]$

In this representation, we can define:

$$a = \underrightarrow{u} \cdot \underrightarrow{v} = \underline{u}^T \underline{v} = \underline{v}^T \underline{u}$$

$$\underrightarrow{w} = \underrightarrow{u} \times \underrightarrow{v} \Rightarrow \underline{w} = \underline{u}^{\times} \underline{v},$$ where

$$\underline{u}^{\times} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

## 5) Important properties

a) $a = \underrightarrow{u} \cdot \underrightarrow{v} = \underrightarrow{v} \cdot \underrightarrow{u} = \underline{u}^T \underline{v} = \underline{v}^T \underline{u}$

b) $\underrightarrow{w} = \underrightarrow{u} \times \underrightarrow{v} = -(\underrightarrow{v} \times \underrightarrow{u})$
$\underline{w} = \underline{u}^{\times} \underline{v} = -\underline{v}^{\times} \underline{u}$

c) Vector triple product

$$\underrightarrow{u} \times (\underrightarrow{v} \times \underrightarrow{w}) = (\underrightarrow{u} \cdot \underrightarrow{w})\,\underrightarrow{v} - (\underrightarrow{u} \cdot \underrightarrow{v})\,\underrightarrow{w}$$
$$\underline{u}^{\times} \underline{v}^{\times} \underline{w} = (\underline{u}^T \underline{w})\,\underline{v} - (\underline{u}^T \underline{v})\,\underline{w}$$

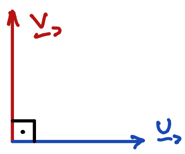
d) Mixed product

$$\underrightarrow{u} \cdot \underrightarrow{v} \times \underrightarrow{w} = \underrightarrow{u} \times \underrightarrow{v} \cdot \underrightarrow{w}$$
$$\underline{u}^T \underline{v}^{\times} \underline{w} = (\underline{u}^{\times} \underline{v})^T \underline{w} = \underline{v}^T \underline{w}^{\times} \underline{u}$$

e) $\underline{u}_{\to} \times (\underline{v}_{\to} + \underline{w}_{\to}) = \underline{u}_{\to} \times \underline{v}_{\to} + \underline{u}_{\to} \times \underline{w}_{\to}$

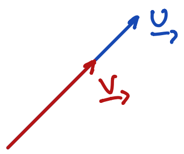
$\underline{u}^{\times} (\underline{v} + \underline{w}) = \underline{u}^{\times} \underline{v} + \underline{u}^{\times} \underline{w}$

f) If $\underline{u}_{\to}$ and $\underline{v}_{\to}$ are perpendicular, then:




$a = \underline{u}_{\to} \cdot \underline{v}_{\to} = 0$

g) If $\underline{u}_{\to}$ and $\underline{v}_{\to}$ are collinears, i.e.:




$\underline{v}_{\to} = \alpha \underline{u}_{\to}$, for $\alpha \in \mathbb{R}$

then:

$\underline{w}_{\to} = \underline{u}_{\to} \times \underline{v}_{\to} = \underline{0}_{\to}$

$\underline{w} = \underline{u}^{\times} \underline{v} = \underline{0}_{3\times 1}$