Vectrices

As mentioned before, we must distinguish between a vector and its representation in the matrix form.

→ Usually, it is represented by a column matrix that is also called "vector".

Hence, we are going to define a notation that will be used to transform a vector in its matrix form and vice-versa.

> We will focus on vector with 3 dimensions. However, the theory can be extended easily for vectors with n-dimensions.

Let $[\hat{\underline{a}}_{>1} \;\; \hat{\underline{a}}_{>2} \;\; \hat{\underline{a}}_{>3}]$ be a set of 3 unit vectors that compose an orthornormal basis of $\mathbb{R}^3$. Hence, any vector $\underline{v}_{>} \in \mathbb{R}^3$ can be written as:

$$\underline{v}_{>} = v_1 \, \underline{a}_{>1} + v_2 \, \underline{a}_{>2} + v_3 \, \underline{a}_{>3}$$

for $v_i \in \mathbb{R}$, $i = [1, 2, 3]$. We define as vectrix the following column matrix that stores the unit vectors of the selected basis:

$$\underline{\mathfrak{F}}_a \triangleq \begin{bmatrix} \underline{a}_{>1} \\ \underline{a}_{>2} \\ \underline{a}_{>3} \end{bmatrix}$$

> Note that the vectrix and the set of unit vectors that contains the basis express the same information. Hence, we can use the same symbol to represent both the vectrix and the basis.

Thus, if the representation of $\underline{v}_{>}$ in the basis $\underline{\mathfrak{F}}_a$ is:

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix},$$

then we can convert between the vector and matrix form of $\underline{v}_{>}$ using:

$$\underline{v}_{>} = \underline{v}^T \underline{\mathfrak{F}}_a = \underline{\mathfrak{F}}_a^T \underline{v}$$

$$\underline{v} = \underline{v}_{>} \cdot \underline{\mathfrak{F}}_a = \underline{\mathfrak{F}}_a \cdot \underline{v}_{>} = \begin{bmatrix} \underline{v}_{>} \cdot \underline{a}_{>1} \\ \underline{v}_{>} \cdot \underline{a}_{>2} \\ \underline{v}_{>} \cdot \underline{a}_{>3} \end{bmatrix}$$

$$\underline{v}^T = \underline{v}_{>} \cdot \underline{\mathfrak{F}}_a^T = \underline{\mathfrak{F}}_a^T \cdot \underline{v}_{>}$$

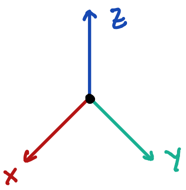
$$\begin{aligned} \underline{v}_{>} \cdot \underline{a}_{>1} &= (v_1 \, \underline{a}_{>1} + v_2 \, \underline{a}_{>2} + v_3 \, \underline{a}_{>3}) \cdot \underline{a}_{>1} \\ &= v_1 \underbrace{\underline{a}_{>1} \cdot \underline{a}_{>1}}_{=1} + v_2 \underbrace{\cancel{\underline{a}_{>2} \cdot \underline{a}_{>1}}}_{0} + v_3 \underbrace{\cancel{\underline{a}_{>3} \cdot \underline{a}_{>1}}}_{0} \\ &= v_1 \end{aligned}$$

Notice that:

$$\underline{\mathcal{F}}_a \times \underline{\mathcal{F}}_a^T = \begin{bmatrix} \hat{\underline{a}}_{\to 1} \times \hat{\underline{a}}_{\to 1} & \underbrace{\hat{\underline{a}}_{\to 1} \times \hat{\underline{a}}_{\to 2}}_{\hat{\underline{a}}_{\to 3}} & \underbrace{\hat{\underline{a}}_{\to 1} \times \hat{\underline{a}}_{\to 3}}_{-\hat{\underline{a}}_{\to 2}} \\ \hat{\underline{a}}_{\to 2} \times \hat{\underline{a}}_{\to 1} & \hat{\underline{a}}_{\to 2} \times \hat{\underline{a}}_{\to 2} & \underbrace{\hat{\underline{a}}_{\to 2} \times \hat{\underline{a}}_{\to 3}}_{\hat{\underline{a}}_{\to 1}} \\ \hat{\underline{a}}_{\to 3} \times \hat{\underline{a}}_{\to 1} & \hat{\underline{a}}_{\to 3} \times \hat{\underline{a}}_{\to 2} & \hat{\underline{a}}_{\to 3} \times \hat{\underline{a}}_{\to 3} \end{bmatrix} = -\underline{\mathcal{F}}_a^{\times}$$

$$\underline{\mathcal{F}}_a \cdot \underline{\mathcal{F}}_a^T = \underline{I}_3$$

This is only true for right-hand reference frame like:



Using that, it is clear the diference between a vector and its matrix form. The former is independent of the selected basis whereas the later only makes sense if a basis is defined. Thus, let $\underline{\mathcal{F}}_a$ and $\underline{\mathcal{F}}_b$ be two basis and let $\underline{v}_{\to}$ be any vector. It can be seen that:

$$\underline{v}_{\to} = \underline{\mathcal{F}}_a^T \underline{v}_a = \underline{\mathcal{F}}_b^T \underline{v}_b,$$

where $\underline{v}_a$ and $\underline{v}_b$ are the representations of $\underline{v}_{\to}$ in $\underline{\mathcal{F}}_a$ and $\underline{\mathcal{F}}_b$, respectively. Analogously:

$$\underline{v}_a = \underline{\mathcal{F}}_a \cdot \underline{v}_{\to}$$
$$\underline{v}_b = \underline{\mathcal{F}}_b \cdot \underline{v}_{\to}$$

Moreover, we can define the previous operations between vectors using vectrizes:

1) $\underline{u}_{\to} \cdot \underline{v}_{\to} \triangleq \underline{u}_a^T \underbrace{\underline{\mathcal{F}}_a \cdot \underline{\mathcal{F}}_a^T}_{\underline{I}_3} \underline{v}_a = \underline{u}_a^T \underline{v}_a$

2) $\underline{u}_{\to} \times \underline{v}_{\to} \triangleq \underline{u}_a^T \underbrace{\underline{\mathcal{F}}_a \times \underline{\mathcal{F}}_a^T}_{-\underline{\mathcal{F}}_a^{\times}} \underline{v}_a = -\underline{u}_a^T \underline{\mathcal{F}}_a^{\times} \underline{v}_a = \underline{\mathcal{F}}_a^T \underline{u}_a^{\times} \underline{v}_a = \underline{\mathcal{F}}_a^T (\underline{u}_a \times \underline{v}_a)$

Proof: $-(\underbrace{\underline{\mathcal{F}}_a^{\times T}}_{-\underline{\mathcal{F}}_a^{\times}} \underline{u}_a)^T \underline{v}_a = (\underline{\mathcal{F}}_a^{\times} \underline{u}_a)^T \underline{v}_a = -(\underline{u}_a^{\times} \underline{\mathcal{F}}_a)^T \underline{v}_a = -\underline{\mathcal{F}}_a^T \underbrace{\underline{u}_a^{\times T}}_{-\underline{u}_a^{\times}} \underline{v}_a = \underline{\mathcal{F}}_a^T \underline{u}_a^{\times} \underline{v}_a = \underline{\mathcal{F}}_a^T (\underline{u}_a \times \underline{v}_a)$

This result is valid for orthonormal, right-hand reference frame, because we used $\underline{\mathcal{F}}_a \times \underline{\mathcal{F}}_a^T = -\underline{\mathcal{F}}_a^{\times}$. Hence, this form to compute the cross product requires that the vectors are represented on the same orthonormal, right-hand frame.