Rotations of Coordinate Frames

It was already shown that we can convert the representation of a vector between two different basis. Let's analyze this procedure in $\mathbb{R}^3$ where there is an important geometric meaning.

Let two distintic basis of $\mathbb{R}^3$ be:

$$\underrightarrow{\mathcal{F}}_a = [\ \underrightarrow{\hat{a}}_1\ \ \underrightarrow{\hat{a}}_2\ \ \underrightarrow{\hat{a}}_3\ ]^T$$

$$\underrightarrow{\mathcal{F}}_b = [\ \underrightarrow{\hat{b}}_1\ \ \underrightarrow{\hat{b}}_2\ \ \underrightarrow{\hat{b}}_3\ ]^T$$

Right-hand and orthornormal

Hence, one can see that:

$$\begin{aligned}
\underrightarrow{\hat{b}}_1 &= C_{11}\, \underrightarrow{\hat{a}}_1 + C_{12}\, \underrightarrow{\hat{a}}_2 + C_{13}\, \underrightarrow{\hat{a}}_3 \\
\underrightarrow{\hat{b}}_2 &= C_{21}\, \underrightarrow{\hat{a}}_1 + C_{22}\, \underrightarrow{\hat{a}}_2 + C_{23}\, \underrightarrow{\hat{a}}_3 \\
\underrightarrow{\hat{b}}_3 &= C_{31}\, \underrightarrow{\hat{a}}_1 + C_{32}\, \underrightarrow{\hat{a}}_2 + C_{33}\, \underrightarrow{\hat{a}}_3
\end{aligned}$$

for $C_{ij} \in \mathbb{R}$, $i \in [1,2,3]$ and $j \in [1,2,3]$. Thus, we can write:

$$\begin{bmatrix} \underrightarrow{\hat{b}}_1 \\ \underrightarrow{\hat{b}}_2 \\ \underrightarrow{\hat{b}}_3 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \cdot \begin{bmatrix} \underrightarrow{\hat{a}}_1 \\ \underrightarrow{\hat{a}}_2 \\ \underrightarrow{\hat{a}}_3 \end{bmatrix}$$

$$\underrightarrow{\mathcal{F}}_b = \underline{C}_{ba} \cdot \underrightarrow{\mathcal{F}}_a$$

The matrix $\underline{C}_{ba}$ is exactly the same matrix we defined previously to change basis. However, it is transposed because of the vectrix definition. Here, we represented the destination basis on the source basis.

Using the vectrices properties, one can see that:

$$\underrightarrow{\mathcal{F}}_b = \underline{C}_{ba}\, \underrightarrow{\mathcal{F}}_a \quad (\cdot\, \underrightarrow{\mathcal{F}}_a^T)$$

$$\underrightarrow{\mathcal{F}}_b \cdot \underrightarrow{\mathcal{F}}_a^T = \underline{C}_{ba}\, \underbrace{\underrightarrow{\mathcal{F}}_a \cdot \underrightarrow{\mathcal{F}}_a^T}_{\underline{I}_3} = \underline{C}_{ba}\, \underline{I}_3 = \underline{C}_{ba}$$

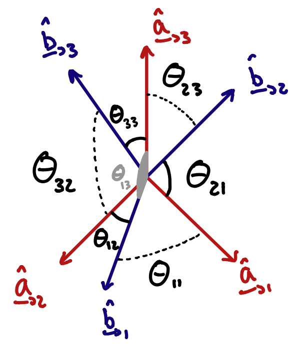
$$\Downarrow$$

$$\underline{C}_{ba} = \underrightarrow{\mathcal{F}}_b \cdot \underrightarrow{\mathcal{F}}_a^T = \begin{bmatrix} \underrightarrow{\hat{b}}_1 \cdot \underrightarrow{\hat{a}}_1 & \underrightarrow{\hat{b}}_1 \cdot \underrightarrow{\hat{a}}_2 & \underrightarrow{\hat{b}}_1 \cdot \underrightarrow{\hat{a}}_3 \\ \underrightarrow{\hat{b}}_2 \cdot \underrightarrow{\hat{a}}_1 & \underrightarrow{\hat{b}}_2 \cdot \underrightarrow{\hat{a}}_2 & \underrightarrow{\hat{b}}_2 \cdot \underrightarrow{\hat{a}}_3 \\ \underrightarrow{\hat{b}}_3 \cdot \underrightarrow{\hat{a}}_1 & \underrightarrow{\hat{b}}_3 \cdot \underrightarrow{\hat{a}}_2 & \underrightarrow{\hat{b}}_3 \cdot \underrightarrow{\hat{a}}_3 \end{bmatrix}$$

Remember that:

$$\underrightarrow{\hat{b}}_i \cdot \underrightarrow{\hat{a}}_j = \overset{=1}{|\underrightarrow{\hat{b}}_i|} \cdot \overset{=1}{|\underrightarrow{\hat{a}}_j|} \cdot \cos(\theta_{ij}) = \cos(\theta_{ij}), \quad i,j \in [1,2,3]$$

where θ_{ij} is the angle between $\underrightarrow{\hat{b}}_i$ and $\underrightarrow{\hat{a}}_j$.



Thus:

$$\underline{C}_{ba} = \begin{bmatrix} \cos\theta_{11} & \cos\theta_{12} & \cos\theta_{13} \\ \cos\theta_{21} & \cos\theta_{22} & \cos\theta_{23} \\ \cos\theta_{31} & \cos\theta_{32} & \cos\theta_{33} \end{bmatrix}$$

This matrix is called **Direction Cosine Matrix** (DCM).

Inverse of direction cosine matrices

From the properties of the vectrizes, we have:

$$\underline{\mathcal{F}}_b \cdot \underline{\mathcal{F}}_b^T = \underline{I}_3$$

Using the DCM:

$$(\underline{C}_{ba}\,\underline{\mathcal{F}}_a)\cdot(\underline{C}_{ba}\,\underline{\mathcal{F}}_a)^T = \underline{I}_3$$

$$\underline{C}_{ba}\,\underbrace{\underline{\mathcal{F}}_a \cdot \underline{\mathcal{F}}_a^T}_{\underline{I}_3}\,\underline{C}_{ba}^T = \underline{I}_3$$

$$\underline{C}_{ba}\,\underline{C}_{ba}^T = \underline{I}_3$$

The construction of the DCM (each line is the representation of 3 orthorgonal vector on the same basis) ensures that it has rank 3. This means that the inverse always exists. Thus:

$$\underline{C}_{ba}\,\underline{C}_{ba}^T = \underline{I}_3 \qquad (\underline{C}_{ba}^{-1}\times)$$

$$\underbrace{\underline{C}_{ba}^{-1}\,\underline{C}_{ba}}_{\underline{I}_3}\,\underline{C}_{ba}^T = \underline{C}_{ba}^{-1}$$

$$\Downarrow$$

$$\underline{C}_{ba}^{-1} = \underline{C}_{ba}^T$$

Provided that the basis are orthornormal!

Thus, the inverse of all DCMs is their transpose.

In the sequence, notice that:

$$\underline{\mathcal{F}}_b = \underline{C}_{ba}\,\underline{\mathcal{F}}_a \overset{(\underline{C}_{ba}\times)}{\Longrightarrow} \underline{C}_{ba}^T\,\underline{\mathcal{F}}_b = \underline{C}_{ba}^T\,\underline{C}_{ba}\,\underline{\mathcal{F}}_a \Longrightarrow \underline{\mathcal{F}}_a = \underline{C}_{ba}^T\,\underline{\mathcal{F}}_b$$

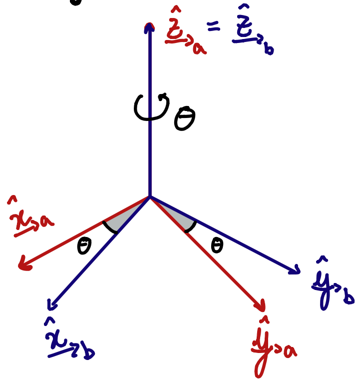
Finally, as expected, $\underline{C}_{ba}^T$ is the matrix that converts the basis b into the basis a.

The representation of a vector in a specific basis is unique. Hence, given the way the DCMs are constructed (by stacking representation of vectors on the same basis), we can conclude that a DCM that transforms the basis a into the basis b is also unique.

Geometric interpretation of the transformation by DCMs

The DCMs, as described here, represent the transformation between two reference frames in $\mathbb{R}^3$ that are orthonormals. It will be shown later that this transformation is always equivalent to a rotation about some axis by an angle (Euler theorem).

Let's consider here, for illustration purposes, two reference frames that are displaced by a single rotation about the z axis:



Using the previous equation, we can compute the matrix that transforms the coordinates of reference frame a to the reference frame b:

$$C_{ba} = \begin{bmatrix} \cos\theta_{11} & \cos\theta_{12} & \cos\theta_{13} \\ \cos\theta_{21} & \cos\theta_{22} & \cos\theta_{23} \\ \cos\theta_{31} & \cos\theta_{32} & \cos\theta_{33} \end{bmatrix}$$

$$C_{ba} = \begin{bmatrix} \cos\theta & \overbrace{\cos(90^\circ-\theta)}^{=\,\mathrm{sen}\,\theta} & \overbrace{\cos 90^\circ}^{=0} \\ \underbrace{\cos(90^\circ+\theta)}_{=-\mathrm{sen}\,\theta} & \cos(\theta) & \cos 90^\circ = 0 \\ \underbrace{\cos 90^\circ}_{0} & \underbrace{\cos 90^\circ}_{0} & \underbrace{\cos 0^\circ}_{1} \end{bmatrix} = \begin{bmatrix} \cos\theta & \mathrm{sen}\,\theta & 0 \\ -\mathrm{sen}\,\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can compute the same matrix for a rotation about the other axes, leading to:

$$\underline{C}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \mathrm{sen}\,\theta \\ 0 & -\mathrm{sen}\,\theta & \cos\theta \end{bmatrix} \rightarrow \text{Rotation about X-axis of an angle } \theta.$$

$$\underline{C}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\mathrm{sen}\,\theta \\ 0 & 1 & 0 \\ \mathrm{sen}\,\theta & 0 & \cos\theta \end{bmatrix} \rightarrow \text{Rotation about Y-axis of an angle } \theta.$$

$$\underline{C}_3(\theta) = \begin{bmatrix} \cos\theta & \mathrm{sen}\,\theta & 0 \\ -\mathrm{sen}\,\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{Rotation about Z-axis of an angle } \theta.$$

For this kind of rotation matrix, we have the following properties:

$$\underline{C}_i^{-1}(\theta) = \underline{C}_i^{T}(\theta) = \underline{C}_i(-\theta), \quad i \in [1,2,3]$$