Operations between vectors and vectrices

We define here some important operations and properties related to vectrices.

Let $\underrightarrow{u}$, $\underrightarrow{v}$, and $\underrightarrow{w}$ $\in \mathbb{R}^3$ be three vectors in a 3D space, and $\underrightarrow{\mathcal{F}}_a$, $\underrightarrow{\mathcal{F}}_b$, and $\underrightarrow{\mathcal{F}}_c$ be three different basis of the same space. Furthermore, assume that the following representations are known:

$$\underline{u}_a = \underrightarrow{\mathcal{F}}_a \cdot \underrightarrow{u}$$
$$\underline{v}_b = \underrightarrow{\mathcal{F}}_b \cdot \underrightarrow{v}$$
$$\underline{w}_c = \underrightarrow{\mathcal{F}}_c \cdot \underrightarrow{w}$$

and that we also have access to the following rotation matrices:

$$\underline{C}_{ab} = \underrightarrow{\mathcal{F}}_a \cdot \underrightarrow{\mathcal{F}}_b^T$$
$$\underline{C}_{ac} = \underrightarrow{\mathcal{F}}_a \cdot \underrightarrow{\mathcal{F}}_c^T$$

> Notice that:
>
> $$\underrightarrow{\mathcal{F}}_b^T = \underrightarrow{\mathcal{F}}_a^T \underline{C}_{ab} \Rightarrow \underrightarrow{\mathcal{F}}_a^T = \underrightarrow{\mathcal{F}}_b^T \underline{C}_{ab}^T$$
> $$\underrightarrow{\mathcal{F}}_c^T = \underrightarrow{\mathcal{F}}_a^T \underline{C}_{ac} \Rightarrow \underrightarrow{\mathcal{F}}_a^T = \underrightarrow{\mathcal{F}}_c^T \underline{C}_{ac}^T$$
>
> Thus:
>
> $$\underrightarrow{\mathcal{F}}_b^T = \underrightarrow{\mathcal{F}}_c^T \underline{C}_{ac}^T \underline{C}_{ab} \Rightarrow \underrightarrow{\mathcal{F}}_c^T = \underrightarrow{\mathcal{F}}_b^T \underline{C}_{ab}^T \underline{C}_{ac}$$

Thus, we have:

1) $$\underrightarrow{u} + \underrightarrow{v} + \underrightarrow{w} = \underrightarrow{\mathcal{F}}_a^T \underline{u}_a + \overbrace{\underrightarrow{\mathcal{F}}_b^T}^{=\underrightarrow{\mathcal{F}}_a^T \underline{C}_{ab}} \underline{v}_b + \overbrace{\underrightarrow{\mathcal{F}}_c^T}^{=\underrightarrow{\mathcal{F}}_a^T \underline{C}_{ac}} \underline{w}_c$$

$$\underrightarrow{u} + \underrightarrow{v} + \underrightarrow{w} = \underrightarrow{\mathcal{F}}_a^T \underline{u}_a + \underrightarrow{\mathcal{F}}_a^T \underline{C}_{ab} \underline{v}_b + \underrightarrow{\mathcal{F}}_a^T \underline{C}_{ac} \underline{w}_c$$
$$\underrightarrow{\mathcal{F}}_a^T \left(\underline{u}_a + \underline{C}_{ab} \underline{v}_b + \underline{C}_{ac} \underline{w}_c\right)$$

↳ matrix operation that computes the representation of $\underrightarrow{u} + \underrightarrow{v} + \underrightarrow{w}$ in the basis $\underrightarrow{\mathcal{F}}_a$.

$$\underrightarrow{u} + \underrightarrow{v} + \underrightarrow{w} = \underrightarrow{\mathcal{F}}_b^T \underline{C}_{ab}^T \underline{u}_a + \underrightarrow{\mathcal{F}}_b^T \underline{v}_b + \underrightarrow{\mathcal{F}}_b^T \underline{C}_{ab}^T \underline{C}_{ac} \underline{w}_c$$
$$\underrightarrow{\mathcal{F}}_b^T \left(\underline{C}_{ab}^T \underline{u}_a + \underline{v}_b + \underline{C}_{ab}^T \underline{C}_{ac} \underline{w}_c\right)$$

↳ matrix operation that computes the representation of $\underrightarrow{u} + \underrightarrow{v} + \underrightarrow{w}$ in the basis $\underrightarrow{\mathcal{F}}_b$.

$$\underrightarrow{u} + \underrightarrow{v} + \underrightarrow{w} = \underrightarrow{\mathcal{F}}_c^T \underline{C}_{ac}^T \underline{u}_a + \underrightarrow{\mathcal{F}}_c^T \underline{C}_{ac}^T \underline{C}_{ab} \underline{v}_b + \underrightarrow{\mathcal{F}}_c^T \underline{w}_c$$
$$\underrightarrow{\mathcal{F}}_c^T \left(\underline{C}_{ac}^T \underline{u}_a + \underline{C}_{ac}^T \underline{C}_{ab} \underline{v}_b + \underline{w}_c\right)$$

↳ matrix operation that computes the representation of $\underrightarrow{u} + \underrightarrow{v} + \underrightarrow{w}$ in the basis $\underrightarrow{\mathcal{F}}_c$.

2) $$\underrightarrow{u} \cdot \underrightarrow{v} = \underline{u}_a^T \overbrace{\underrightarrow{\mathcal{F}}_a \cdot \underrightarrow{\mathcal{F}}_b^T}^{\underline{C}_{ab}} \underline{v}_b = \underbrace{\underline{u}_a^T \underline{C}_{ab} \underline{v}_b}_{\text{All representations are converted to } \underrightarrow{\mathcal{F}}_a.} = \underbrace{\left(\underline{C}_{ab}^T \underline{u}_a\right)^T \underline{v}_b}_{\text{All representations are converted to } \underrightarrow{\mathcal{F}}_b.}$$

3) $\underrightarrow{u} \times \underrightarrow{v} = \underline{u}_a^T \underline{\mathcal{F}}_a \times \overset{= \underline{\mathcal{F}}_a^T C_{ab}}{\underline{\mathcal{F}}_b^T} \underline{v}_b = \underline{u}_a^T \overset{= -\underline{\mathcal{F}}_a^\times}{\boxed{\underline{\mathcal{F}}_a \times \underline{\mathcal{F}}_a^T}} C_{ab} \underline{v}_b = -\underline{u}_a^T \underline{\mathcal{F}}_a^\times C_{ab} \underline{v}_b = \underline{\mathcal{F}}_a^T \overset{(*)}{\underline{u}_a^\times C_{ab} \underline{v}_b} = \underline{\mathcal{F}}_a^T (\underline{u}_a \times C_{ab} \underline{v}_b)$

$$-\underline{u}_a^T \underline{\mathcal{F}}_a^\times = -(\underbrace{\underline{\mathcal{F}}_a^{\times T}}_{= -\underline{\mathcal{F}}_a^\times \text{ (see the definition)}} \underline{u}_a)^T = (\underline{\mathcal{F}}_a^\times \underline{u}_a)^T = (-\underline{u}_a^\times \underline{\mathcal{F}}_a)^T = -\underline{\mathcal{F}}_a^T \underbrace{\underline{u}_a^{\times T}}_{= -\underline{u}_a^\times} = \underline{\mathcal{F}}_a^T \underline{u}_a^\times$$

All representations are converted to the basis $\underline{\mathcal{F}}_a$

$\underrightarrow{u} \times \underrightarrow{v} = \underline{u}_a^T \overset{= C_{ab} \underline{\mathcal{F}}_b}{\underline{\mathcal{F}}_a} \times \underline{\mathcal{F}}_b^T \underline{v}_b = \overset{(C_{ab}^T \underline{u}_a)^T}{\underline{u}_a^T C_{ab}} \overset{= -\underline{\mathcal{F}}_b^\times}{\boxed{\underline{\mathcal{F}}_b \times \underline{\mathcal{F}}_b^T}} \underline{v}_b = -\underline{(C_{ab}^T \underline{u}_a)^T \underline{\mathcal{F}}_b^\times} \underline{v}_b = \underline{\mathcal{F}}_b^T \overset{(o)}{(C_{ab}^T \underline{u}_a)^\times \underline{v}_b} = \underline{\mathcal{F}}_b^T [(C_{ab}^T \underline{u}_a) \times \underline{v}_b]$

use the previous result letting $C_{ab}^T \underline{u}_a = \underline{u}_b$

All representations are converted to the basis $\underline{\mathcal{F}}_b$.

Notice that, replacing $\underline{\mathcal{F}}_a^T = \underline{\mathcal{F}}_b^T C_{ab}^T$ in (*) and comparing to (o), leads to:

$$\underline{\mathcal{F}}_a^T \underline{u}_a^\times C_{ab} \underline{v}_b = \underline{\mathcal{F}}_b^T (C_{ab}^T \underline{u}_a)^\times \underline{v}_b$$
$$\underline{\mathcal{F}}_b^T C_{ab}^T \underline{u}_a^\times C_{ab} \underline{v}_b = \underline{\mathcal{F}}_b^T (C_{ab}^T \underline{u}_a)^\times \underline{v}_b$$

Since the representation of a vector in a basis is unique, then:

$$C_{ab}^T \underline{u}_a^\times C_{ab} \underline{v}_b = (C_{ab}^T \underline{u}_a)^\times \underline{v}_b$$

Finally, since this is true $\forall \underline{v}_b \in \mathbb{R}^3$, then:

$$C_{ab}^T \underline{u}_a^\times C_{ab} = (C_{ab}^T \underline{u}_a)^\times$$

In the following, there are some additional operations between vetrices and vectors.

Let $\underline{\mathcal{F}}_a = [\hat{\underrightarrow{a}}_1 \; \hat{\underrightarrow{a}}_2 \; \hat{\underrightarrow{a}}_3]^T$, $\underline{\mathcal{F}}_b = [\hat{\underrightarrow{b}}_1 \; \hat{\underrightarrow{b}}_2 \; \hat{\underrightarrow{b}}_3]^T$, and $\underrightarrow{v} \in \mathbb{R}^3$. Thus, we have:

1) Dot product between a vectrix and a vector:

$\underrightarrow{v} \cdot \underline{\mathcal{F}}_a = \underline{\mathcal{F}}_a \cdot \underrightarrow{v} = \underline{v}_a$ => column matrix

2) Cross product between a vectrix and a vector:

$$\underrightarrow{v} \times \underline{\mathcal{F}}_a \triangleq \begin{bmatrix} \underrightarrow{v} \times \hat{\underrightarrow{a}}_1 \\ \underrightarrow{v} \times \hat{\underrightarrow{a}}_2 \\ \underrightarrow{v} \times \hat{\underrightarrow{a}}_3 \end{bmatrix} = -\underline{v}_a^\times \underline{\mathcal{F}}_a = \underline{\mathcal{F}}_a^\times \underline{v}_a = \underline{\mathcal{F}}_c \Rightarrow \text{new vectrix}$$

A new basis only if $\underrightarrow{v}$ is not aligned with $\hat{\underrightarrow{a}}_1$, $\hat{\underrightarrow{a}}_2$, or $\hat{\underrightarrow{a}}_3$.
If $\underrightarrow{v}$ is not unitary, then the new basis will be orthogonal but the vector in the basis will not be unitary.

3) Scalar product between vectrices:

$$\underrightarrow{\mathcal{F}}_a \cdot \underrightarrow{\mathcal{F}}_b^T \triangleq \begin{bmatrix} \hat{\underrightarrow{a}}_1 \cdot \hat{\underrightarrow{b}}_1 & \hat{\underrightarrow{a}}_1 \cdot \hat{\underrightarrow{b}}_2 & \hat{\underrightarrow{a}}_1 \cdot \hat{\underrightarrow{b}}_3 \\ \underrightarrow{a}_2 \cdot \underrightarrow{b}_1 & \underrightarrow{a}_2 \cdot \underrightarrow{b}_2 & \underrightarrow{a}_2 \cdot \underrightarrow{b}_3 \\ \underrightarrow{a}_3 \cdot \underrightarrow{b}_1 & \underrightarrow{a}_3 \cdot \underrightarrow{b}_2 & \underrightarrow{a}_3 \cdot \underrightarrow{b}_3 \end{bmatrix} = \underline{C}_{ab} \implies 3 \times 3 \text{ matrix } \left(\mathbb{R}^{3\times 3}\right)$$

$$\underrightarrow{\mathcal{F}}_a^T \cdot \underrightarrow{\mathcal{F}}_b = \hat{\underrightarrow{a}}_1 \cdot \hat{\underrightarrow{b}}_1 + \hat{\underrightarrow{a}}_2 \cdot \hat{\underrightarrow{b}}_2 + \hat{\underrightarrow{a}}_3 \cdot \hat{\underrightarrow{b}}_3 \implies \text{scalar } (\mathbb{R})$$

4) Cross product between vectrizes

$$\underrightarrow{\mathcal{F}}_a \times \underrightarrow{\mathcal{F}}_b^T \triangleq \begin{bmatrix} \hat{\underrightarrow{a}}_1 \times \hat{\underrightarrow{b}}_1 & \hat{\underrightarrow{a}}_1 \times \hat{\underrightarrow{b}}_2 & \hat{\underrightarrow{a}}_1 \times \hat{\underrightarrow{b}}_3 \\ \underrightarrow{a}_2 \times \underrightarrow{b}_1 & \underrightarrow{a}_2 \times \underrightarrow{b}_2 & \underrightarrow{a}_2 \times \underrightarrow{b}_3 \\ \underrightarrow{a}_3 \times \underrightarrow{b}_1 & \underrightarrow{a}_3 \times \underrightarrow{b}_2 & \underrightarrow{a}_3 \times \underrightarrow{b}_3 \end{bmatrix} = -\underrightarrow{\mathcal{F}}_a^{\times} \underline{C}_{ab}$$

$$\hookrightarrow \underrightarrow{\mathcal{F}}_a \times \underrightarrow{\mathcal{F}}_b^T = \underrightarrow{\mathcal{F}}_a \times \underrightarrow{\mathcal{F}}_a^T \underline{C}_{ab} = -\underrightarrow{\mathcal{F}}_a^{\times} \underline{C}_{ab}$$

5) Product between a vectrix and a column matrix

$$\underline{v}_a^T \underrightarrow{\mathcal{F}}_a = \underrightarrow{\mathcal{F}}_a^T \underline{v}_a = \underrightarrow{v} \implies \text{vector } (\mathbb{R}^3)$$