Dyadics

A dyadic is the result of a product between two vectors:

$$\underrightarrow{D} = \underrightarrow{u}\,\underrightarrow{v}$$

This operation provides a description for important properties. For example, we will see that the inertia matrix of a rigid body can be represented by a dyadic. Thus, the dynamics equations can be written in a vector form.

In the following, we will see some properties of the dyadics.

Properties

1) This product is not commutative.

$$\underrightarrow{u}\,\underrightarrow{v} \neq \underrightarrow{v}\,\underrightarrow{u}$$

2) The dot product between a vector and a dyadic is a vector:

$$\underrightarrow{w} = \underrightarrow{h} \cdot \underrightarrow{D}$$

$$\hookrightarrow \underrightarrow{h} \cdot \underrightarrow{D} = \overbrace{\underrightarrow{h} \cdot \underrightarrow{u}}^{\text{scalar} = \lambda}\,\underrightarrow{v} = \lambda\,\underrightarrow{v} = \underrightarrow{w}$$

This operation is also not commutative:

$$\underrightarrow{h} \cdot \underrightarrow{D} \neq \underrightarrow{D} \cdot \underrightarrow{h}$$

$$\hookrightarrow \underrightarrow{D} \cdot \underrightarrow{h} = \underrightarrow{u}\,\overbrace{\underrightarrow{v} \cdot \underrightarrow{h}}^{\text{scalar} = \alpha} = \alpha\,\underrightarrow{u}$$

$$\hookrightarrow \underrightarrow{h} \cdot \underrightarrow{D} = \underbrace{\underrightarrow{h} \cdot \underrightarrow{u}}_{\text{scalar} = \lambda}\,\underrightarrow{v} = \lambda\,\underrightarrow{v}$$

3) We have the following definitions:

$$\underrightarrow{J} = (\underrightarrow{w} \times \underrightarrow{u})\,\underrightarrow{v} = \underrightarrow{w} \times \underrightarrow{u}\,\underrightarrow{v} \triangleq \underrightarrow{w} \times \underrightarrow{D}$$

$$\underrightarrow{K} = \underrightarrow{u}\,(\underrightarrow{v} \times \underrightarrow{w}) = \underrightarrow{u}\,\underrightarrow{v} \times \underrightarrow{w} \triangleq \underrightarrow{D} \times \underrightarrow{w}$$

In general, $\underrightarrow{J} \neq \underrightarrow{K}$.

Matrix representation of dyadics

Let $\underrightarrow{u}$ and $\underrightarrow{v}$ be two vectors in $\mathbb{R}^3$ with representations $\underline{u}_a$ and $\underline{v}_a$ in the basis $\underrightarrow{\mathcal{F}}_a$. Thus:

$$\underrightarrow{D} = \underrightarrow{u}\,\underrightarrow{v} = \underrightarrow{\mathcal{F}}_a^T\,\underline{u}_a\,\underline{v}_a^T\,\underrightarrow{\mathcal{F}}_a = \underrightarrow{\mathcal{F}}_a^T\,\underline{D}_a\,\underrightarrow{\mathcal{F}}_a$$

Thus, the representation of the dyadics is a 3×3 matrix such as:

$$\underline{D}_a = \underline{u}_a \underline{v}_a^T$$

The matrix form can be obtained from the vectorial form using:

$$\underline{D}_a = \underline{\mathcal{F}}_a \cdot \underset{\to}{D} \cdot \underline{\mathcal{F}}_a^T$$

$$\hookrightarrow \underline{\mathcal{F}}_a \cdot \underset{\to}{D} \cdot \underline{\mathcal{F}}_a^T = \underline{\mathcal{F}}_a \cdot \underset{\to}{u}\, \underset{\to}{v} \cdot \underline{\mathcal{F}}_a^T = \overbrace{\underline{\mathcal{F}}_a \cdot \underline{\mathcal{F}}_a^T}^{=I} \underline{u}_a \underline{v}_a^T \overbrace{\underline{\mathcal{F}}_a \cdot \underline{\mathcal{F}}_a^T}^{=I} = \underline{u}_a \underline{v}_a^T = \underline{D}_a$$

The dot product between a dyadic and two vectors results in a scalar:

$$\underset{\to}{u} \cdot \underset{\to}{D} \cdot \underset{\to}{v} = \underline{u}_a^T \overbrace{\underline{\mathcal{F}}_a \cdot \underline{\mathcal{F}}_a^T}^{=I} \underline{D}_a \overbrace{\underline{\mathcal{F}}_a \cdot \underline{\mathcal{F}}_a^T}^{=I} \underline{v}_a = \underbrace{\underline{u}_a^T}_{1\times 3} \underbrace{\underline{D}_a}_{3\times 3} \underbrace{\underline{v}_a}_{3\times 1} \Rightarrow 1\times 1 \text{ (scalar)}$$

A null dyadics is a dyadic in which its representation is a null matrix in $\mathbb{R}^{3\times 3}$:

$$\underset{\to}{D} = \underset{\to}{0} \Rightarrow \underline{D}_a = \underline{0}_{3\times 3}$$

A unitary dyadics is a dyadic in which its representation is an identity matrix in $\mathbb{R}^{3\times 3}$:

$$\underset{\to}{D} = \underset{\to}{1} \Rightarrow \underline{D}_a = \underline{I}_{3\times 3}$$

In this case:

$$\underset{\to}{u} = \underset{\to}{u} \cdot \underset{\to}{1} = \underset{\to}{1} \cdot \underset{\to}{u}$$

$$\hookrightarrow \underline{\mathcal{F}}_a^T\, \underline{I}\, \overbrace{\underline{\mathcal{F}}_a \cdot \underline{\mathcal{F}}_a^T}^{=I} \underline{u}_a = \underline{\mathcal{F}}_a^T \underline{u}_a = \underset{\to}{u}$$

Representation of dyadics in different basis

Given a dyadic $\underset{\to}{D}$ with representation $\underline{D}_a$ in the basis $\underline{\mathcal{F}}_a$, we can find its representation in the basis $\underline{\mathcal{F}}_b$ using:

$$\underline{D}_a = \underline{\mathcal{F}}_a \cdot \underset{\to}{D} \cdot \underline{\mathcal{F}}_a^T \Rightarrow \underline{\mathcal{F}}_a^T \underline{D}_a \underline{\mathcal{F}}_a = \underset{\to}{D}$$

$$\underline{D}_b = \underline{\mathcal{F}}_b \cdot \underset{\to}{D} \cdot \underline{\mathcal{F}}_b^T$$

$$\Downarrow$$

$$\underline{D}_b = \underbrace{\underline{\mathcal{F}}_b \cdot \underline{\mathcal{F}}_a^T}_{=\underline{C}_{ba}} \underline{D}_a \underbrace{\underline{\mathcal{F}}_a \cdot \underline{\mathcal{F}}_b^T}_{=\underline{C}_{ab} = \underline{C}_{ba}^T} = \underline{C}_{ba} \underline{D}_a \underline{C}_{ab} = \underline{C}_{ba} \underline{D}_a \underline{C}_{ba}^T$$

$$\underline{D}_b = \underline{C}_{ba} \underline{D}_a \underline{C}_{ab} = \underline{C}_{ba} \underline{D}_a \underline{C}_{ba}^T$$

If the vectors $\underset{\to}{u}$ and $\underset{\to}{v}$ are known in different basis, then:

$$\underset{\to}{D} = \underset{\to}{u}\, \underset{\to}{v} = \underline{\mathcal{F}}_a^T \underline{u}_a \underline{v}_b^T \underline{\mathcal{F}}_b = \underline{\mathcal{F}}_a^T \underline{D}_{ab} \underline{\mathcal{F}}_b$$

Analogously:

$$\underline{D}_{ab} = \underline{u}_a \underline{v}_b^T = \underline{\mathcal{F}}_a \cdot \underset{\to}{u}\, \underset{\to}{v} \cdot \underline{\mathcal{F}}_b^T = \underline{\mathcal{F}}_a \cdot \underset{\to}{D} \cdot \underline{\mathcal{F}}_b^T$$

If it is desired to represent the dyadic entirely in a single basis, then:

$$\underline{D}_a = \underline{\mathcal{F}}_a \cdot \underline{D} \cdot \underline{\mathcal{F}}_a^T = \underbrace{\boxed{\underline{\mathcal{F}}_a \cdot \underline{\mathcal{F}}_a^T}}_{\underline{I}} \underline{D}_{ab} \underbrace{\boxed{\underline{\mathcal{F}}_b \cdot \underline{\mathcal{F}}_a^T}}_{\underline{C}_{ba} = \underline{C}_{ab}^T} = \underline{D}_{ab} \underline{C}_{ab}^T$$

$$\underline{D}_b = \underline{\mathcal{F}}_b \cdot \underline{D} \cdot \underline{\mathcal{F}}_b^T = \underbrace{\boxed{\underline{\mathcal{F}}_b \cdot \underline{\mathcal{F}}_a^T}}_{\underline{C}_{ba} = \underline{C}_{ab}^T} \underline{D}_{ab} \underline{\mathcal{F}}_a \cdot \underline{\mathcal{F}}_a^T = \underline{C}_{ab}^T \underline{D}_{ab}$$