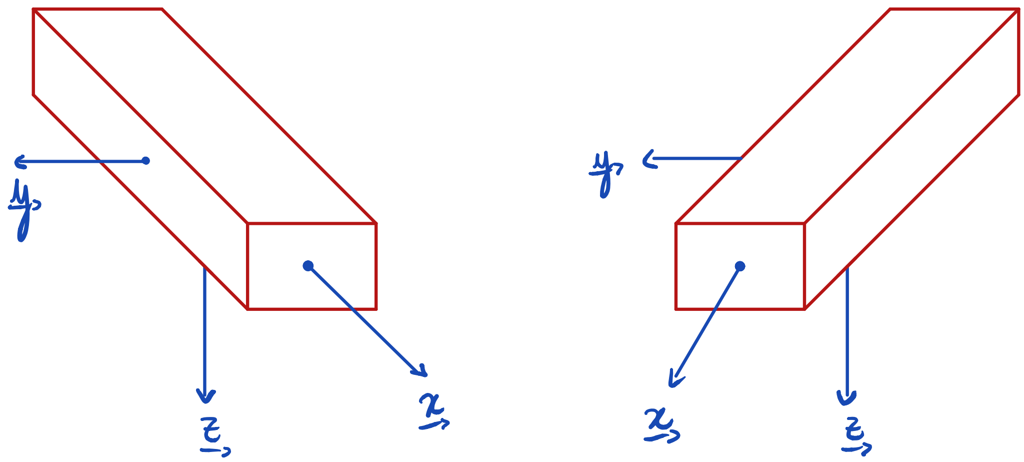
Attitude representation

Attitude: the attitude of a rigid body is its orientation representation in space.

First, we need to select a reference system that is fixed to the body, i.e., it will move and rotate together with the body:



Thus, we can represent how this reference system relates with other known reference system that is not fixed to the body. This is sufficient to determine the attitude of the object.

Usually, the known, external reference system is an "inertial" reference system, i.e., it does not have angular motion.

→ Notice that, depending on the application, a reference system with sufficiently small angular motion can be considered inertial for the sake of simplification.

If the body reference system is represented by $\mathcal{F}_b$ and the inertial system (reference) is represent by $\mathcal{F}_i$, then the attitude representation is obtained finding a relationship between $\mathcal{F}_b$ and $\mathcal{F}_i$.

Direction cosine matrix (DCM)

As we saw previously, DCMs are used to transform the basis of a vector representation. Thus, they can be used to represent the attitude of an object.

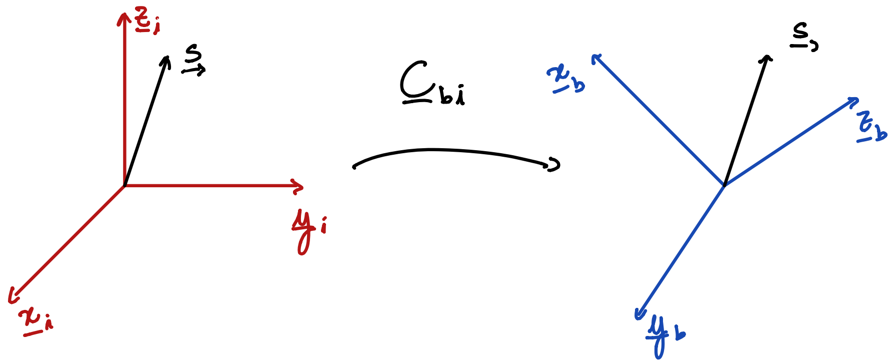
> Given the body coordinate frame $\mathcal{F}_b$ and the reference coordinate frame $\mathcal{F}_i$, if we can compute the matrix $\underline{C}_{bi}$ every instant, then we successfully describe the attitude of the object.

Note: In this case, if we know the representation of a vector in the inertial system $\mathcal{F}_i$ (like the sun position, for example), then we can obtain its representation in the body reference frame using:

$$\underline{v}_b = \underline{C}_{bi}\, \underline{v}_i$$

Example: suppose that, in a simulation, we want to obtain the component of the solar vector in the direction normal to the solar panel to compute the power generation.
Let:

- $\underline{\hat{s}}_i$ be the representation of the sun unitary vector in the inertial reference system.
- $\underline{C}_{bi}$ be the matrix that transform the body coordinate system to the inertial reference system.
- $\underline{\hat{n}}_b$ be the vector aligned with the solar panel normal vector.



First, we need to obtain the representation of the sun unitary vetor in the body reference frame:

$$\underline{\hat{s}}_b = \underline{C}_{bi}\,\underline{\hat{s}}_i$$

Thus, the projection of this vector in the solar panel is:

$$\cos\theta = \underline{\hat{n}}_b^T\,\underline{\hat{s}}_b \longrightarrow \text{The generated energy is proportional to } \cos\theta.$$

Composition of rotations

Let $\underline{\mathfrak{F}}_a$, $\underline{\mathfrak{F}}_b$, and $\underline{\mathfrak{F}}_c$ be three coordinate systems where we Know $\underline{C}_{ab}$ and $\underline{C}_{bc}$. Thus:

$$\underline{v}_a = \underline{C}_{ab}\,\underline{v}_b$$
$$\underline{v}_b = \underline{C}_{bc}\,\underline{v}_c$$

If we want to obtain the matrix that transforms the system $\underline{\mathfrak{F}}_c$ in the system $\underline{\mathfrak{F}}_a$, then:

$$\underline{v}_a = \underline{C}_{ab}\,\underline{C}_{bc}\,\underline{v}_c = \underline{C}_{ac}\,\underline{v}_c$$

This result is valid $\forall\ \underline{v} \in \mathbb{R}^3$.

$$\Downarrow$$

$$\underline{C}_{ac} = \underline{C}_{ab}\,\underline{C}_{bc}$$

The matrix $\underline{C}_{ac}$ was obtained using a sequence of rotations. Notice that the order of the rotations has influence in the result, because the matrix multiplication is not commutative.

Redundant parameters in the DCMs

The DCMs were constructed using the representation of three orthonormal vectors in another reference system. This means that we need 9 numbers to represent a DCM. However, since we are restricting the coordinate systems to be right-handed and orthonormals, then the same information requires only three parameters.

Let $\mathcal{F}_a = [\hat{a}_{>1}\ \hat{a}_{>2}\ \hat{a}_{>3}]^T$ and $\mathcal{F}_b = [\hat{b}_{>1}\ \hat{b}_{>2}\ \hat{b}_{>3}]^T$ be two right-handed, orthonormal reference systems. Thus, consider the following information:

- The direction of $\hat{b}_{>1}$ in $\mathcal{F}_a$ expressed in spherical coordinates (θ and ϕ).
- The projection of $\hat{b}_{>3}$ in $\hat{a}_{>3}$. In this case, it cannot be 0!

Hence, the DCM can be computed using:

$$\hat{b}_{>1} = \operatorname{sen}\phi \cdot \cos\theta\ \hat{a}_{>1} + \operatorname{sen}\phi \operatorname{sen}\theta\ \hat{a}_{>2} + \cos\phi\ \hat{a}_{>3}$$

$$\hat{b}_{>3} = b_{3,1}\ \hat{a}_{>1} + b_{3,2}\ \hat{a}_{>2} + \underbrace{\boxed{b_{3,3}}}_{\text{Known}}\ \hat{a}_{>3}$$

We know that:

$$\hat{b}_{>3} \cdot \hat{b}_{>1} = 0 \Rightarrow b_{3,1} \operatorname{sen}\phi \cos\theta + b_{3,2} \operatorname{sen}\phi \operatorname{sen}\theta + b_{3,3}\cos\phi = 0$$

$$\hat{b}_{>3} \cdot \hat{b}_{>3} = 1 \Rightarrow b_{3,1}^2 + b_{3,2}^2 + b_{3,3}^2 = 1$$

Here we used the property that the system is orthonormal.

System with two equations and two variables ($b_{3,1}$ and $b_{3,2}$).

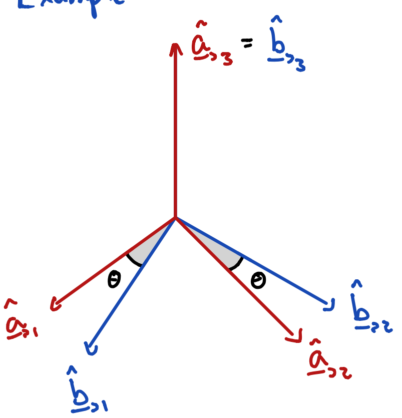
Solving this system, we obtain $b_{3,1}$ and $b_{3,2}$. Thus, we now have $\hat{b}_{>1}$ and $\hat{b}_{>3}$ represented in $\mathcal{F}_a$. Finally, we can find $\hat{b}_{>2}$ using:

$$\hat{b}_{>2} = \hat{b}_{>3} \times \hat{b}_{>1}$$

Here we used the properties that the system is orthonormal and right-handed.

Hence, it is now possible to compute the matrix $\underline{C}_{ba}$.

Example



In this case:

$\theta \neq 0°$

$\phi = 90°$

$b_{3,3} = 1$

Thus:

$$\hat{b}_{>1} = \overset{=1}{\operatorname{sen}\phi}\cos\theta\ \hat{a}_{>1} + \overset{=1}{\operatorname{sen}\phi}\operatorname{sen}\theta\ \hat{a}_{>2} + \overset{0}{\cancel{\cos\phi}}\ \hat{a}_{>3}$$

$$= \cos\theta\ \hat{a}_{>1} + \operatorname{sen}\theta\ \hat{a}_{>2}$$

$$\hat{b}_{>3} = b_{3,1}\ \hat{a}_{>1} + b_{3,2}\ \hat{a}_{>2} + b_{3,3}\ \hat{a}_{>3}$$

$$\Downarrow$$

$$\left.\begin{aligned} b_{3,1}\cos\theta + b_{3,2}\operatorname{sen}\theta = 0 \\ b_{3,1}^2 + b_{3,2}^2 = 0 \end{aligned}\right\} \quad b_{3,1} = b_{3,2} = 0$$

$$\hat{\underline{b}}_{>2} = \hat{\underline{b}}_{>3} \times \hat{\underline{b}}_{>1}$$

$$\hat{\underline{b}}_{>2} = \begin{vmatrix} \hat{\underline{a}}_{>1} & \hat{\underline{a}}_{>2} & \hat{\underline{a}}_{>3} \\ 0 & 0 & 1 \\ \cos\theta & \operatorname{sen}\theta & 0 \end{vmatrix} = -\operatorname{sen}\theta\, \hat{\underline{a}}_{>1} + \cos\theta\, \hat{\underline{a}}_{>2}$$

Finally:

$$\underline{C}_{ba} = \begin{bmatrix} \cos\theta & \operatorname{sen}\theta & 0 \\ -\operatorname{sen}\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This means that with only 3 parameters we could compute the entire DCM.

It is important to mention that those 3 parameters are only one possible set of an infinite number of possibilities. Furthermore, those parameters cannot be used in all cases, e.g., they fail if $b_{3,3} = 0$.

Euler proved that only 3 parameters are necessary to unequivocally define the spatial orientation of a reference system with respect to another one.

Thus, since a DCM has 9 parameters, we conclude that there are high redundancy in this representation. In the following, we will see more efficient representations for the attitude of an object.