Euler angle and axis

Euler proved that any two reference systems in $\mathbb{R}^3$ can be aligned by a single rotation about a specific axis. The axis in which the reference system must be rotated to align with the other one is called Euler axis and the angle of the rotation is called Euler angle.

In the following, we prove this statement using DCMs. → In this case, let's threat $\underline{C}_{ba}$ as a linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$.

Let $\underline{C}_{ba}$ be the DCM that transforms the reference system $\mathcal{F}_a$ into $\mathcal{F}_b$.

Preliminary: Determinant of a DCM

To prove the Euler's theorem, we need to compute the determinant of the DCM. Since a DCM is orthonormal, then:

$$\underline{C}_{ba}^{-1} = \underline{C}_{ba}^{T}$$

This leads to:

$$\underline{C}_{ba}^{-1} \cdot \underline{C}_{ba} = \underline{I}_3$$
$$\underline{C}_{ba}^{T} \cdot \underline{C}_{ba} = \underline{I}_3$$
$$\Downarrow$$
$$\det(\underline{C}_{ba}^{T} \cdot \underline{C}_{ba}) = \det(\underline{I}_3)$$
$$\det(\underline{C}_{ba}^{T})\det(\underline{C}_{ba}) = 1 \longrightarrow \det(\underline{A}\underline{B}) = \det(\underline{A}).\det(\underline{B}) \text{ if } \underline{A} \text{ and } \underline{B} \text{ are square matrices.}$$
$$\det(\underline{C}_{ba})\det(\underline{C}_{ba}) = 1 \longrightarrow \det(\underline{A}) = \det(\underline{A}^{T})$$
$$\det(\underline{C}_{ba})^2 = 1$$
$$\det(\underline{C}_{ba}) = \pm 1$$

We saw that the matrix $\underline{C}_{ba}$ is composed by the vectors in $\mathcal{F}_b$ represented in $\mathcal{F}_a$. In $\mathbb{R}^{3\times3}$, the determinant can be computed using:

$$\underline{C}_{ba} = \begin{bmatrix} \underline{b}_{1a}^{T} \\ \underline{b}_{2a}^{T} \\ \underline{b}_{3a}^{T} \end{bmatrix} \Rightarrow \det(\underline{C}_{ba}) = +\underline{b}_{1a}^{T}(\underline{b}_{2a} \times \underline{b}_{3a})$$

If $\mathcal{F}_a$ and $\mathcal{F}_b$ are right-handed, then $\underline{b}_{2a} \times \underline{b}_{3a} = \underline{b}_{1a}$. Thus:

$$\det(\underline{C}_{ba}) = \underline{b}_{1a}^{T}\underline{b}_{1a} = +1 \quad \Rightarrow \quad \det(\underline{C}_{ba}) = 1$$

It is known that if the determinant of a linear transformation is negative, then it reflects an odd number of axes. In other words, it changes the orientation of the reference system. However, since we are assuming right-handed coordinate frames, then we do not have reflections.

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1) Rotation axis

We need to prove two statements to show that exists a rotation axis (Euler axis):

a) There is an axis that any vector aligned with it has the same representation in both systems.
b) The plane perpendicular to that axis is mapped to itself.
c) The angle between vectors in the plane perpendicular to that axis are preserved after the transformation.

Proving (a)

If there is a rotation axis, then the representations of a vector $\underline{v}$, aligned to that axis in both systems are equals:

$$\underline{v}_a = \underline{v}_b = \underline{v}$$

Thus:

$$
\begin{aligned}
\underline{C}_{ba}\,\underline{v}_a &= \underline{v}_b\\
\underline{C}_{ba}\,\underline{v} &= \underline{v}\\
\underline{C}_{ba}\,\underline{v} &= \underline{I}_3\,\underline{v}\\
(\underline{C}_{ba}-\underline{I}_3)\,\underline{v} &= \underline{0}_{3\times1}
\end{aligned}
$$

This equation has a nontrivial solution if, and only if, $(\underline{C}_{ba}-\underline{I}_3)$ is singular. We know that a matrix is singular if, and only if, its determinant is 0. Thus:

$$
\begin{aligned}
\det(\underline{C}_{ba}-\underline{I}_3) &= \overset{=1}{\det[\underline{C}_{ba}^T]}\,\det(\underline{C}_{ba}-\underline{I}_3)\\
&= \det[\underline{C}_{ba}^T(\underline{C}_{ba}-\underline{I}_3)] && \longrightarrow \det(\underline{A})\det(\underline{B}) = \det(\underline{A}\,\underline{B})\\
&= \det(\underbrace{\underline{C}_{ba}^T\underline{C}_{ba}}_{\underline{I}_3} - \underline{C}_{ba}^T)\\
&= \det(\underline{I}_3 - \underline{C}_{ba}^T)\\
&= \det(\underline{I}_3^T - \underline{C}_{ba}^T) && \longrightarrow \underline{I}_3 = \underline{I}_3^T\\
&= \det[(\underline{I}_3 - \underline{C}_{ba})^T] && \longrightarrow \underline{A}^T - \underline{B}^T = (\underline{A}-\underline{B})^T\\
&= \det(\underline{I}_3 - \underline{C}_{ba}) && \longrightarrow \det(A^T) = \det(A)\\
&= \det[-(\underline{C}_{ba}-\underline{I}_3)]\\
&= -\det(\underline{C}_{ba}-\underline{I}_3) && \longrightarrow \det(-A) = -\det(A)
\end{aligned}
$$

Thus:

$$\det(\underline{C}_{ba}-\underline{I}_3) = -\det(\underline{C}_{ba}-\underline{I}_3) \Rightarrow \det(\underline{C}_{ba}-\underline{I}_3) = 0$$

Hence, we proved that exists $\underline{v} \neq \underline{0}_{3\times1}$ in which:

$$\underline{C}_{ba}\,\underline{v} = \underline{v}$$

This means that $\lambda = 1$ is an eigenvalue of $\underline{C}_{ba}$ associated to the eigenvector $\underline{v}$.

Finally, we concluded that exists an invariant axis in which all vector aligned with this axis have the same representation in both coordinate systems $\mathcal{F}_a$ and $\mathcal{F}_b$.

Proving (b)

Let $\underline{u}$ be a vector perpendicular to $\underline{v}$, that is aligned with the rotation axis of $\underline{C}_{ba}$ when represented in $\underline{\mathcal{F}}_a$. Thus:

$$\underline{u}_a^T \underline{v}_a = \underline{0}_{3\times1}$$

Furthermore:

$$\begin{aligned}\underline{u}_b^T \underline{v}_b &= (\underline{C}_{ba}\underline{u}_a)^T \underline{C}_{ba}\underline{v}_a \\ &= \underline{u}_a^T \underbrace{\underline{C}_{ba}\underline{C}_{ba}^T}_{\underline{I}_3} \underline{v}_a = \underline{u}_a^T \underline{v}_a = \underline{0}_{3\times1}\end{aligned}$$

This is trivial when we remember that $\underline{C}_{ba}$ is a linear transformation that only changes the basis between two orthonormal, right-handed coordinate systems.

Thus, the plane perpendicular to $\underline{v}$ continues perpendicular to $\underline{v}$ after the transformation $\underline{C}_{ba}$. Since $\underline{v}$ is invariant to this transformation, so it is this plane.

Proving (c)

Let $\underline{u}_a$ and $\underline{w}_a$ two vectors in the plane perpendicular to $\underline{v}$. Thus:

$$\underline{u}_a^T \underline{w}_a = |\underline{u}_a||\underline{w}_a|\cos\theta$$

Moreover:

$$\begin{aligned}\underline{u}_b^T \underline{w}_b &= (\underline{C}_{ba}\underline{u}_a)^T(\underline{C}_{ba}\underline{w}_a) \\ &= \underline{u}_a^T \underbrace{\underline{C}_{ba}^T \underline{C}_{ba}}_{\underline{I}_3} \underline{w}_a = \underline{u}_a^T \underline{w}_a \\ &= |\underline{u}_a||\underline{w}_a|\cos\theta\end{aligned}$$

Since $\underline{C}_{ba}$ is orthonormal:

$$|\underline{u}_a| = |\underline{u}_b|$$
$$|\underline{w}_a| = |\underline{w}_b|$$

Thus:

$$\underline{u}_b^T \underline{w}_b = \underline{u}_a^T \underline{w}_a \Rightarrow \cancel{|\underline{u}_b|}\,\cancel{|\underline{w}_b|}\cos\theta^* = \cancel{|\underline{u}_a|}\,\cancel{|\underline{w}_a|}\cos\theta \Rightarrow \cos\theta^* = \cos\theta$$

θ: angle between $\underline{u}$, and $\underline{w}$, before the transformation

θ^*: angle between $\underline{u}$, and $\underline{w}$, after the transformation.

This means that after the transformation $\underline{C}_{ba}$ the angles between the vectors in the plane perpendicular to $\underline{v}$ are kept constants.

This is trivial when we remember that $\underline{C}_{ba}$ is a linear transformation that only changes the basis between two orthonormal, right-handed coordinate systems.

Finally, vectors aligned to $\underline{v}$ do not change their representation after the transformation $\underline{C}_{ba}$. Vectors perpendicular to $\underline{v}$ keep the same angle between them after the transformation and continue perpendicular to $\underline{v}$. Thus, we conclude that <u>every</u> DCM represents a rotation of the coordinate system about a specific axis.

Finding the rotation axis

The direction of the rotation axis is the same as the direction of the eigenvector associated to the eigenvalue 1. To compute this direction, let:

$$\underline{C}_{ba} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

If a vector $\underline{v} \neq \underline{0}_{3\times1}$ is aligned to the rotation axis, then:

$$(\underline{C}_{ba} - \underline{I}_3)\,\underline{v} = \underline{0}_{3\times1}$$

Notice that $\underline{v}$ is also an eigenvector of $\underline{C}_{ba}^T$:

$$\underline{C}_{ba}\,\underline{v} = \underline{v} \qquad (\underline{C}_{ba}^T \ast)$$
$$\underbrace{\underline{C}_{ba}^T\,\underline{C}_{ba}}_{\underline{I}_3}\,\underline{v} = \underline{C}_{ba}^T\,\underline{v}$$
$$\underline{C}_{ba}^T\,\underline{v} = \underline{v}$$

Thus, we have:

$$(\underline{C}_{ba} - \underline{C}_{ba}^T)\,\underline{v} = \underline{C}_{ba}\,\underline{v} - \underline{C}_{ba}^T\,\underline{v} = \underline{v} - \underline{v} = \underline{0}_{3\times1}$$

Let $\underline{Q} \triangleq \underline{C}_{ba} - \underline{C}_{ba}^T$. Notice that this matrix has the following property:

$$\underline{Q}^T = (\underline{C}_{ba} - \underline{C}_{ba}^T)^T = \underline{C}_{ba}^T - \underline{C}_{ba} = -(\underline{C}_{ba} - \underline{C}_{ba}^T) = -\underline{Q}$$
$$\underline{Q}^T = -\underline{Q}$$

Thus, we can write:

$$\underline{Q} = \begin{bmatrix} 0 & q_3 & -q_2 \\ -q_3 & 0 & q_1 \\ q_2 & -q_1 & 0 \end{bmatrix}$$

where:

$$q_1 = C_{23} - C_{32}$$
$$q_2 = C_{31} - C_{13}$$
$$q_3 = C_{12} - C_{21}$$

Furthermore:

$$(\underline{C}_{ba} - \underline{C}_{ba}^T)\underline{v} = \underline{Q}\,\underline{v} = \underline{0}_{3\times1}$$

leading to the following linear system:

$$\begin{bmatrix} 0 & q_3 & -q_2 \\ -q_3 & 0 & q_1 \\ q_2 & -q_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $q \triangleq [q_1 \; q_2 \; q_3]^T$. Thus, the above linear system can be written as:

$$-q^{\times}\underline{v} = -\underline{q} \times \underline{v} = \underline{0}_{3\times1}$$

Finally, it turns out that $\underline{v}$ must be aligned with q, i.e.:

$$\underline{v} = \alpha \underline{q}, \quad \forall \alpha \in \mathbb{R}$$

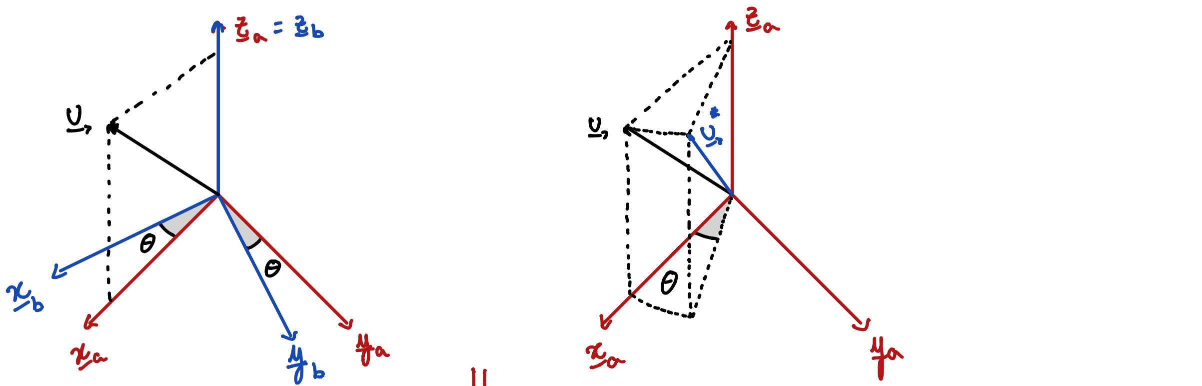
Usually, it is desired that $\underline{v}$ be a unitary vector. Thus, the rotation axis direction of the DCM $\underline{C}_{ba}$ is:

$$\underline{v} = \frac{\underline{q}}{|\underline{q}|}, \quad \underline{q} = \begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix}$$

2) Rotation angle

Here we will use the theory of <u>linear transformations</u> to compute the Euler angle using a geometric approach.

As we saw, a DCM can be intrepreted as a linear transformation that rotates a coordinate system about a specific axis. However, we can also intrepret a DCM as a linear transformation that rotates the vectors in the same coordinate system.



$$\Downarrow$$

$$\underline{v}_b = \underline{v}_a^*$$

<u>Conclusion</u>: the representation of a vector ($\underline{v}_b$) in a coordinate system obtained by rotating $\theta°$ about an axis $\underline{e}$ is equal to the representation of another vector ($\underline{v}_a^*$) in the original coordinate system $\underline{\mathcal{F}}_a$ obtained by rotating the vector $\underline{v}_s$ $-\theta°$ about the same axis $\underline{e}$.

A rotation of vectors about the $\underline{x}$-axis is given by the following linear transformation:

$$\underline{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Thus, let $\underline{D}$ be the basis transformation that transform the coordinate system $\mathcal{F}_a$ into any other that its $\underline{x}$-axis is aligned with the rotation axis of the DCM $\underline{C}_{ba}$. Hence, given the previous discussion and for any $\underline{v}_a \in \mathbb{R}^3$, we have:

$$\underline{v}_a^* = \underline{D}^T \, \underline{R}_x(-\theta) \, \underline{D} \, \underline{v}_a$$

→ Transforms the representation to that system that its $\underline{x}$-axis is aligned with the rotation axis of the DCM $\underline{C}_{ba}$.

→ Rotates the vector by $-\theta°$, where θ is the Euler angle of $\underline{C}_{ba}$.

→ Converts the representation back to $\mathcal{F}_a$.

This leads to:

$$\underline{v}_b = \underline{v}_a^* \quad \Rightarrow \quad \underline{C}_{ba}\,\underline{v}_a = \underline{D}^T \underline{R}_x(-\theta)\,\underline{D}\,\underline{v}_a$$

Since this equality is valid $\forall\, \underline{v}_a \in \mathbb{R}^3$, then:

$$\underline{C}_{ba} = \underline{D}^T \underline{R}_x(-\theta)\,\underline{D}$$

Hence:

$$\begin{aligned} tr(\underline{C}_{ba}) &= tr\left(\underline{D}^T \underline{R}_x(-\theta)\,\underline{D}\right) \\ &= tr\left(\underline{R}_x(-\theta)\,\underbrace{\underline{D}\,\underline{D}^T}_{\underline{I}_3}\right) \longrightarrow tr(\underline{A}\,\underline{B}) = tr(\underline{B}\,\underline{A}) \\ &= tr\left(\underline{R}_x(-\theta)\right) \end{aligned}$$

$$tr(\underline{C}_{ba}) = tr\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} = 2\cos\theta + 1$$

Finally:

$$\cos\theta = \frac{tr(\underline{C}_{ba}) - 1}{2}$$

$$\cos\theta = \frac{1}{2}\left(C_{11} + C_{22} + C_{33} - 1\right)$$

3) Rotation direction

Here we will define the rotation direction and also find an expression for $|q|$. Notice that:

$$\underline{Q} = (\underline{C}_{ba} - \underline{C}_{ba}^T) = (\underline{D}^T \underline{R}_x(-\theta)\,\underline{D}) - (\underline{D}^T \underline{R}_x(-\theta)\,\underline{D})^T$$
$$= \underline{D}^T \underline{R}_x(-\theta)\,\underline{D} - \underline{D}^T \underline{R}_x^T(-\theta)\,\underline{D} \longrightarrow (\underline{A}\underline{B}\underline{C})^T = \underline{C}^T\underline{B}^T\underline{A}^T$$
$$= \underline{D}^T [\underline{R}_x(-\theta) - \underline{R}_x^T(-\theta)]\,\underline{D}$$

$$\underline{Q} = \underline{D}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2\sin\theta \\ 0 & -2\sin\theta & 0 \end{bmatrix} \underline{D}$$

Let $\underline{r} = [2\sin\theta \;\; 0 \;\; 0]^T$, then:

$$\overset{= -q^\times}{\underline{Q}} = \underline{D}^T(-\underline{r}^\times)\,\underline{D}$$
$$-\underline{q}^\times = -\underline{D}^T \underline{r}^\times \underline{D}$$
$$\underline{q}^\times = \underline{D}^T \underline{r}^\times \underline{D}$$
$$\underline{q}^\times = (\underline{D}^T\underline{r})^\times$$
$$\Downarrow$$
$$\underline{q} = \underline{D}^T\underline{r}$$

Exercise 2.1-d: $(\underline{C}_{ab}^T \underline{U})^\times = \underline{C}_{ab}^T \underline{U}^\times \underline{C}_{ab}$

$$\Downarrow$$

$$\underline{D}^T \underline{r}^\times \underline{D} = (\underline{D}^T\underline{r})^\times$$

$\longrightarrow$ This means that $\underline{q}$ and $\underline{r}$ are representations of <u>the same vector</u> in different coordinate systems.

$\to$ $\underline{r}$ is a vector aligned with the rotation axis represented in the selected coordinate system in which its $\underline{x}$-axis is aligned with the same rotation axis (Euler axis).

$\to$ The positive rotation direction is obtained by the direction of $\underline{q}$ using the right-hand rule.

Thus, we have:

$$|q| = \sqrt{\underline{q}^T\underline{q}} = \sqrt{\underline{r}^T \underbrace{\underline{D}\,\underline{D}^T}_{\underline{I}_3} \underline{r}} = \sqrt{\underline{r}^T\underline{r}} = \sqrt{4\sin^2\theta} = |2\sin\theta| \longrightarrow |q| = 2\sin\theta \;\text{ if }\; 0 \le \theta \le 180°.$$

Assuming that the rotation direction about $\underline{q}$ follows the right-hand rule, then:

$$\underline{v} = \begin{bmatrix} \dfrac{C_{23} - C_{32}}{2\sin\theta} \\[2ex] \dfrac{C_{31} - C_{13}}{2\sin\theta} \\[2ex] \dfrac{C_{12} - C_{21}}{2\sin\theta} \end{bmatrix}, \text{ assuming } \sin\theta \neq 0! \implies \underline{v}^\times = \frac{1}{2\sin\theta}(\underline{C}_{ba}^T - \underline{C}_{ba})$$

This solution is unique if we limit:

$0° \le \theta \le 180°$ $\longrightarrow$ Notice that $(\underline{v}, \theta) = (-\underline{v}, -\theta) = (-\underline{v}, 360° - \theta)$

$\to$ The singularity when $\theta = 180°$ will be handled in the following.

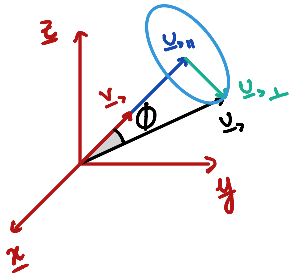
$\to$ If $\theta = 0°$, then no rotation happened. In this case, the Euler axis is undefined.

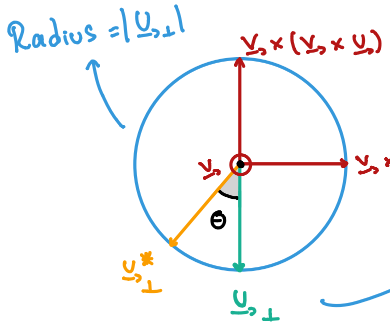
Relationship between DCMs and Euler angle and axis

We will show in the following that if the DCM $\underline{C}_{ba}$ has the Euler axis $\underline{v}$ and the Euler angle θ, then we can write:

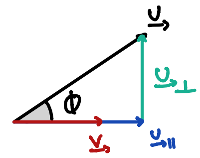
$$\underline{M} \triangleq \cos\theta\, \underline{I}_3 + (1-\cos\theta)\underline{v}\,\underline{v}^T - \sin\theta\, \underline{v}^\times = \underline{C}_{ba}$$

Every vector $\vec{u}$ can be decomposed into a part parallel to $\vec{v}$ and a part perpendicular to $\vec{v}$:





$$\vec{u} = \vec{u}_{\parallel} + \vec{u}_{\perp}$$
$$\vec{u}^* = \vec{u}_{\parallel} + \vec{u}^*_{\perp}$$



Notice that

$|\vec{u}_{\perp}| = |\vec{u}| \sin\phi$

$|\vec{v} \times \vec{u}| = |\vec{u}|\underbrace{|\vec{v}|}_{=1} \sin\phi = |\vec{u}| \sin\phi = |\vec{u}_{\perp}|$

$|\vec{v} \times (\vec{v} \times \vec{u})| = \underbrace{|\vec{v}|}_{=1}|\vec{v} \times \vec{u}| \underbrace{\sin 90^\circ}_{=1} = |\vec{v} \times \vec{u}| = |\vec{u}_{\perp}|$

The vector $\vec{u}^*$ that is obtained by rotating $\vec{u}$ about the Euler vector $\vec{v}$ by $-\theta$ can be obtained as follows:

$$\vec{u}^* = \vec{u}_{\parallel} + \vec{u}^*_{\perp} = (\vec{v} \cdot \vec{u})\vec{v} - \sin\theta\, \vec{v} \times \vec{u} - \cos\theta\, \vec{v} \times (\vec{v} \times \vec{u})$$

We showed that $\underline{u}^*_a = \underline{u}_b$, thus:

$$\begin{aligned} \underline{u}_b = \underline{u}^*_a &= (\underline{v} \cdot \underline{u}_a)\underline{v} - \sin\theta\, \underline{v}^\times \underline{u}_a - \cos\theta\, \underline{v}^\times \underline{v}^\times \underline{u}_a \longrightarrow \text{Remember that } \underline{v}_a = \underline{v}_b = \underline{v} \\ &= \underline{v}(\underline{v} \cdot \underline{u}_a) - \sin\theta\, \underline{v}^\times \underline{u}_a - \cos\theta\, (\underline{v}\,\underline{v}^T - \underline{I}_3)\underline{u}_a \longrightarrow \underline{v}^\times \underline{v}^\times = \underline{v}\,\underline{v}^T - \underline{I}_3 \\ &= \underline{v}\,\underline{v}^T \underline{u}_a - \sin\theta\, \underline{v}^\times \underline{u}_a - \cos\theta\, \underline{v}\,\underline{v}^T \underline{u}_a + \cos\theta\, \underline{u}_a \\ &= [\cos\theta\, \underline{I}_3 + (1-\cos\theta)\underline{v}\,\underline{v}^T - \sin\theta\, \underline{v}^\times]\underline{u}_a \end{aligned}$$

Hence, we have:

$$\underline{u}_b = [\cos\theta\, \underline{I}_3 + (1-\cos\theta)\underline{v}\,\underline{v}^T - \sin\theta\, \underline{v}^\times]\underline{u}_a$$
$$\underline{u}_b = \underline{C}_{ba}\underline{u}_a$$

Since those equalities are valid $\forall\, \vec{u} \in \mathbb{R}^3$, then:

$$\underline{M} \triangleq \cos\theta\, \underline{I}_3 + (1-\cos\theta)\underline{v}\,\underline{v}^T - \sin\theta\, \underline{v}^\times = \underline{C}_{ba}$$

Singularity when $\theta = 180°$

Now we can find an expression for the Euler angle when $\theta = 180°$. Assuming that $\underline{v} = [v_1\ v_2\ v_3]^T$:

$$\underline{C}_{ba} = \underbrace{\cos 180°}_{=-1}\, \underline{I}_3 + (1 - \underbrace{\cos 180°}_{=-1})\underline{v}\,\underline{v}^T - \cancelto{0}{\sin 180°}\, \underline{v}^\times = -\underline{I}_3 + 2\underline{v}\,\underline{v}^T$$

$$\underline{C}_{ba} = \begin{bmatrix} 2v_1^2-1 & 2v_1v_2 & 2v_1v_3 \\ 2v_1v_2 & 2v_2^2-1 & 2v_1v_3 \\ 2v_1v_3 & 2v_1v_3 & 2v_3^2-1 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \longrightarrow \underline{C}_{ba} = 2\underline{v}\,\underline{v}^T - \underline{I}_3$$

Finally, if $\theta = 180°$, then:

$$V_1 = \pm\sqrt{\frac{1 + C_{11}}{2}} \qquad V_2 = \pm\sqrt{\frac{1 + C_{22}}{2}} \qquad V_3 = \pm\sqrt{\frac{1 + C_{33}}{2}}$$

where the signs can be obtained from:

$$2V_1V_2 = C_{12} = C_{21}$$
$$2V_1V_3 = C_{13} = C_{31}$$
$$2V_2V_3 = C_{23} = C_{32}$$

In this case, notice that:
$\underline{C}_{ba} = \underline{v}\,\underline{v}^T - \underline{I}_3$

Bonus: $\underline{M}$ is indeed an orthonormal matrix

$$\underline{M}\,\underline{M}^T = \left[\cos\theta\, \underline{I}_3 + (1-\cos\theta)\underline{v}\,\underline{v}^T - \sin\theta\, \underline{v}^\times\right]\left[\cos\theta\, \underline{I}_3 + (1-\cos\theta)\underline{v}\,\underline{v}^T - \sin\theta\, \underline{v}^\times\right]^T$$

$$= \left[\cos\theta\, \underline{I}_3 + (1-\cos\theta)\underline{v}\,\underline{v}^T - \sin\theta\, \underline{v}^\times\right]\left[\cos\theta\, \underline{I}_3 + (1-\cos\theta)\underline{v}\,\underline{v}^T - \sin\theta\, \underline{v}^{\times T}\right] \longrightarrow (\underline{A}+\underline{B})^T = \underline{A}^T + \underline{B}^T$$

$$= \left[\cos\theta\, \underline{I}_3 + (1-\cos\theta)\underline{v}\,\underline{v}^T - \sin\theta\, \underline{v}^\times\right]\left[\cos\theta\, \underline{I}_3 + (1-\cos\theta)\underline{v}\,\underline{v}^T + \sin\theta\, \underline{v}^\times\right] \longrightarrow \underline{v}^{\times T} = -\underline{v}^\times$$

$$= \cos^2\theta\, \underline{I}_3 + (\cos\theta - \cos^2\theta)\underline{v}\,\underline{v}^T - \cos\theta\sin\theta\, \underline{v}^\times$$
$$(\cos\theta - \cos\theta^2)\underline{v}\,\underline{v}^T + (1-\cos\theta)^2\underline{v}\,\underline{v}^T\underline{v}\,\underline{v}^T + (\sin\theta\cos\theta - \sin\theta)\underline{v}^\times\underline{v}\,\underline{v}^T$$
$$+ \sin\theta\cos\theta\, \underline{v}^\times + (\sin\theta - \sin\theta\cos\theta)\,\underline{v}\underbrace{\underline{v}^T\underline{v}^\times} - \sin^2\theta\, \underline{v}^\times\underline{v}^\times$$

$$\begin{cases} \underline{v}^T\underline{v} = 1 \\ \underline{v}^\times\underline{v} = \underline{0}_{3\times1} \\ \underline{v}^\times\underline{v}^\times = \underline{v}\,\underline{v}^T - \underline{I}_3 \end{cases} \qquad (\underline{v}^{\times T}\underline{v}) = -(\underline{v}^\times\underline{v})$$

$$= \cos^2\theta\, \underline{I}_3 + \cancel{\cos\theta}\,\underline{v}\,\underline{v}^T - \cos^2\theta\,\underline{v}\,\underline{v}^T - \cancel{\cos\theta\sin\theta}\,\underline{v}^\times + \cancel{\cos\theta}\,\underline{v}\,\underline{v}^T - \cancel{\cos^2\theta}\,\underline{v}\,\underline{v}^T + \underline{v}\,\underline{v}^T - 2\cancel{\cos\theta}\,\underline{v}\,\underline{v}^T$$
$$+ \cancel{\cos^2\theta}\,\underline{v}\,\underline{v}^T + \cancel{\sin\theta\cos\theta}\,\underline{v}^\times - \sin^2\theta\,(\underline{v}\,\underline{v}^T - \underline{I}_3)$$

$$= \cos^2\theta\, \underline{I}_3 - \cos^2\theta\, \underline{v}\,\underline{v}^T + \underline{v}\,\underline{v}^T - \sin^2\theta\, \underline{v}\,\underline{v}^T + \sin\theta^2\, \underline{I}_3$$

$$= \underbrace{\left[\cos^2\theta + \sin^2\theta\right]}_{=1}\underline{I}_3 + \underline{v}\,\underline{v}^T - \underbrace{\left[\sin^2\theta + \cos^2\theta\right]}_{=1}\underline{v}\,\underline{v}^T$$

$$= \underline{I}_3 + \underline{v}\,\underline{v}^T - \underline{v}\,\underline{v}^T = \underline{I}_3$$

Thus, $\underline{M}\,\underline{M}^T = \underline{I}_3$, which means that $\underline{M}$ is orthonormal.

Conclusion: we showed that the same information in the DCM can be described by the Euler axis and the Euler angle. Thus, only 3 parameters are necessary.

Composing rotations with Euler angle and axis

Supose that we Know the Euler angle and axis $(\underline{v}_1, \theta_1)$ that transforms $\underline{\mathcal{F}}_a$ into $\underline{\mathcal{F}}_b$ and the Euler angle and axis $(\underline{v}_2, \theta_2)$ that transforms $\underline{\mathcal{F}}_b$ into $\underline{\mathcal{F}}_c$. Using that information, we show in the following how to obtain $(\underline{v}_3, \theta_3)$ that transforms $\underline{\mathcal{F}}_a$ into $\underline{\mathcal{F}}_c$ (composed rotation).

We Know that:

$$\underline{C}_{ca} = \underline{C}_{cb}\,\underline{C}_{ba} = \left[c_2 \underline{I}_3 + (1-c_2)\underline{v}_2\underline{v}_2^T - s_2\underline{v}_2^{\times} \right]\left[c_1 \underline{I}_3 + (1-c_1)\underline{v}_1\underline{v}_1^T - s_1\underline{v}_1^{\times} \right]$$

where: $\underline{c}_1 = \cos\theta_1$

$\underline{c}_2 = \cos\theta_2$

Thus, we Know that the Euler angle of $\underline{C}_{ca}$ can be computed using:

$$2c_3 + 1 = tr\,\underline{C}_{ca} = tr\left\{ \left[c_2 \underline{I}_3 + (1-c_2)\underline{v}_2\underline{v}_2^T - s_2\underline{v}_2^{\times} \right]\left[c_1 \underline{I}_3 + (1-c_1)\underline{v}_1\underline{v}_1^T - s_1\underline{v}_1^{\times} \right] \right\}$$

$$= tr\Big[c_1c_2 I_3 + (c_2 - c_1c_2)\underline{v}_1\underline{v}_1^T - s_1c_2\underline{v}_1^{\times} + (c_1 - c_1c_2)\underline{v}_2\underline{v}_2^T + (1-c_1)(1-c_2)\underline{v}_2\underline{v}_2^T\underline{v}_1\underline{v}_1^T + (s_1c_2 - s_1)\underline{v}_2\underline{v}_2^T\underline{v}_1^{\times}$$
$$- c_1s_2\underline{v}_2^{\times} + (c_1s_2 - s_2)\underline{v}_2^{\times}\underline{v}_1\underline{v}_1^T + s_1s_2\underline{v}_2^{\times}\underline{v}_1^{\times} \Big]$$

↳ Using that $tr(\alpha\underline{A} + \beta\underline{B}) = \alpha\, tr(\underline{A}) + \beta\, tr(\underline{B})$

$$2c_3 + 1 = c_1c_2\, tr(\underline{I}_3) + (c_2 - c_1c_2)\, tr(\underline{v}_1\underline{v}_1^T) - s_1c_2\, tr(\underline{v}_1^{\times}) + (c_1 - c_1c_2)\, tr(\underline{v}_2\underline{v}_2^T) + (1-c_1)(1-c_2)\, tr(\underline{v}_2\underline{v}_2^T\underline{v}_1\underline{v}_1^T)$$
$$+ (s_1c_2 - s_1)\, tr(\underline{v}_2\underline{v}_2^T\underline{v}_1^{\times}) - c_1s_2\, tr(\underline{v}_2^{\times}) + (c_1s_2 - s_2)\, tr(\underline{v}_2^{\times}\underline{v}_1\underline{v}_1^T) + s_1s_2\, tr(\underline{v}_2^{\times}\underline{v}_1^{\times})$$

↳ If $\underline{v}_i = [x \;\; y \;\; z]$, $i \in [1,2]$, then:

$$tr(\underline{v}_i^{\times}) = tr\begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix} = 0 \qquad tr(\underline{v}_i\underline{v}_i^T) = tr\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} = x^2 + y^2 + z^2 = 1$$

$$2c_3 + 1 = 3c_1c_2 + c_2 - c_1c_2 + c_1 - c_1c_2 + (1 - c_1 - c_2 + c_1c_2)\, tr(\underline{v}_2\underline{v}_2^T\underline{v}_1\underline{v}_1^T) + (s_1c_2 - s_1)\, tr(\underline{v}_2\underline{v}_2^T\underline{v}_1^{\times})$$
$$+ (c_1s_2 - s_2)\, tr(\underline{v}_2^{\times}\underline{v}_1\underline{v}_1^T) + s_1s_2\, tr(\underline{v}_2^{\times}\underline{v}_1^{\times})$$

↳ Using that $tr(\underline{A}^T\underline{B}) = tr(\underline{B}\underline{A}^T)$

$$tr(\underline{v}_2\underline{v}_2^T\underline{v}_1\underline{v}_1^T) = tr(\underbrace{\underline{v}_1^T\underline{v}_2}\,\underline{v}_2^T\underline{v}_1) = tr(c_\gamma^2) = c_\gamma^2$$

$\cos\gamma = c_\gamma$, where γ is the angle between $\underline{v}_1$ and $\underline{v}_2$

Notice that

$$tr(\underline{v}_2\underline{v}_2^T\underline{v}_1^{\times}) \overset{tr(AB)=tr(BA)}{=} tr(\underline{v}_1^{\times}\underline{v}_2\underline{v}_2^T) = tr\left[(\underline{v}_1 \times \underline{v}_2)\underline{v}_2^T \right] \overset{tr(AB^T)=tr(B^TA)}{=} tr\left[\underline{v}_2^T \underbrace{(\underline{v}_1 \times \underline{v}_2)}_{\text{Perpendicular to } \underline{v}_2} \right] = tr(0) = 0$$

$$tr(\underline{v}_2^{\times}\underline{v}_1\underline{v}_1^T) = tr\left[(\underline{v}_2 \times \underline{v}_1)\underline{v}_1^T \right] = tr\left[\underline{v}_1^T \underbrace{(\underline{v}_2 \times \underline{v}_1)}_{\text{perpendicular to } \underline{v}_1} \right] = tr(0) = 0$$

$$2C_3+1 = C_1C_2 + C_1 + C_2 + (1-C_1-C_2+C_1C_2)C_\gamma^2 + S_1S_2\,\mathrm{tr}(\underline{v}_2^\times \underline{v}_1^\times)$$

$\hookrightarrow$ Using $\underline{v}_1 = [x_1 \;\; y_1 \;\; z_1]^T$ and $\underline{v}_2 = [x_2 \;\; y_2 \;\; z_2]^T$, then:

$$\mathrm{tr}(\underline{v}_2^\times \underline{v}_1^\times) = \mathrm{tr}\left\{\begin{bmatrix} 0 & z_1 & -y_1 \\ -z_1 & 0 & x_1 \\ y_1 & -x_1 & 0 \end{bmatrix}\begin{bmatrix} 0 & z_2 & -y_2 \\ -z_2 & 0 & x_2 \\ y_2 & -x_2 & 0 \end{bmatrix}\right\} = \mathrm{tr}\begin{bmatrix} -z_1z_2-y_1y_2 & \cdot & \cdot \\ \cdot & -z_1z_2-x_1x_2 & \cdot \\ \cdot & \cdot & -y_1y_2-x_1x_2 \end{bmatrix}$$

$$= -2z_1z_2 - 2y_1y_2 - 2x_1x_2 = -2(\underline{v}_1^T\underline{v}_2) = -2\cos\gamma = -2C_\gamma$$

$$2C_3+1 = C_1C_2 + C_1 + C_2 + (1-C_1-C_2+C_1C_2)C_\gamma^2 + S_1S_2(-2C_\gamma)$$
$$= C_1C_2 + C_1 + C_2 + (1-C_1)(1-C_2)C_\gamma^2 - 2S_1S_2C_\gamma$$

Thus, we have:

$$2\cos\theta_3 + 1 = \cos\theta_1\cos\theta_2 + \cos\theta_1 + \cos\theta_2 + (1-\cos\theta_1)(1-\cos\theta_2)\cos^2\gamma - 2\sin\theta_1\sin\theta_2\cos\gamma$$

Using half-angles, we can obtain a simpler expression. Remember that:

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 \qquad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

Thus:

$$2\cos\theta_3 + 1 = 2\left[2\cos^2\frac{\theta_3}{2} - 1\right] + 1 = 4\cos^2\frac{\theta_3}{2} - 1$$

$$\cos\theta_1\cos\theta_2 + \cos\theta_1 + \cos\theta_2 = \left(2\cos^2\frac{\theta_1}{2} - 1\right)\left(2\cos^2\frac{\theta_2}{2} - 1\right) + 2\cos^2\frac{\theta_1}{2} - 1 + 2\cos^2\frac{\theta_2}{2} - 1$$
$$= 4\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} \cancel{- 2\cos^2\frac{\theta_1}{2}} \cancel{- 2\cos^2\frac{\theta_2}{2}} \cancel{+1} \cancel{+ 2\cos^2\frac{\theta_1}{2}} \cancel{-1} \cancel{+ 2\cos^2\frac{\theta_2}{2}} - 1$$
$$= 4\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} - 1$$

$$(1-\cos\theta_1)(1-\cos\theta_2)\cos^2\gamma = \left(2 - 2\cos^2\frac{\theta_1}{2}\right)\left(2 - 2\cos^2\frac{\theta_2}{2}\right)\cos^2\gamma = 4\overbrace{\left(1-\cos^2\frac{\theta_1}{2}\right)}^{\sin^2\frac{\theta_1}{2}}\overbrace{\left(1-\cos^2\frac{\theta_2}{2}\right)}^{\sin^2\frac{\theta_2}{2}}\cos^2\gamma$$
$$= 4\sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}\cos^2\gamma$$

$$-2\sin\theta_1\sin\theta_2\cos\gamma = -8\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_2}{2}\cos\gamma$$

Using those results, one can see that:

$$4\cos^2\frac{\theta_3}{2} \cancel{-1} = 4\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} \cancel{-1} - 8\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\gamma + 4\sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}\cos^2\gamma \quad (\div 4)$$

$$\cos^2\frac{\theta_3}{2} = \cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} - 2\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\gamma + \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}\cos^2\gamma$$

$$\cos^2\frac{\theta_3}{2} = \left[\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\Upsilon\right]^2$$

$$\cos\frac{\theta_3}{2} = \pm\left(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\Upsilon\right)$$

To remove this ambiguity, notice that $\theta_3 = 0$ if $\theta_1 = \theta_2 = 0$. Thus:

$$\cos\frac{\theta_3}{2} = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\Upsilon$$

The Euler axis $\underline{a}_3$ can be computed using that:

$$\underline{v}_3^{\times} = \frac{1}{2\sin\theta_3}\left(\underline{C}_{ca}^T - \underline{C}_{ca}\right), \quad \text{if } \operatorname{sen}\theta_3 \neq 0.$$

$$\begin{aligned}
2\sin\theta_3\,\underline{v}_3^{\times} = {} & \underline{C}_{ca}^T - \underline{C}_{ca} = \\
& [C_1C_2 I_3 + (C_2 - C_1C_2)\underline{v}_1\underline{v}_1^T - S_1C_1\underline{v}_1^{\times} + (C_1 - C_1C_2)\underline{v}_2\underline{v}_2^T + (1-C_1)(1-C_2)\underline{v}_2\underline{v}_2^T\underline{v}_1\underline{v}_1^T + (S_1C_2 - S_1)\underline{v}_2\underline{v}_2^T\underline{v}_1^{\times} \\
& -C_1S_2\underline{v}_2^{\times} + (C_1S_2 - S_2)\underline{v}_2^{\times}\underline{v}_1\underline{v}_1^T + S_1S_2\underline{v}_2^{\times}\underline{v}_1^{\times}]^T - \\
& [C_1C_2 I_3 + (C_2 - C_1C_2)\underline{v}_1\underline{v}_1^T - S_1C_1\underline{v}_1^{\times} + (C_1 - C_1C_2)\underline{v}_2\underline{v}_2^T + (1-C_1)(1-C_2)\underline{v}_2\underline{v}_2^T\underline{v}_1\underline{v}_1^T + (S_1C_2 - S_1)\underline{v}_2\underline{v}_2^T\underline{v}_1^{\times} \\
& -C_1S_2\underline{v}_2^{\times} + (C_1S_2 - S_2)\underline{v}_2^{\times}\underline{v}_1\underline{v}_1^T + S_1S_2\underline{v}_2^{\times}\underline{v}_1^{\times}]
\end{aligned}$$

$$\begin{aligned}
2\sin\theta_3\,\underline{v}_3^{\times} = {} & \cancel{C_1C_2 I_3} + (C_2 - \cancel{C_1}C_2)\underline{v}_1\underline{v}_1^T + S_1C_2\underline{v}_1^{\times} + (C_1 - \cancel{C_1C_2})\underline{v}_2\underline{v}_2^T + (1-C_1)(1-C_2)\underline{v}_1\underline{v}_1^T\underline{v}_2\underline{v}_2^T - (S_1C_2 - S_2)\underline{v}_1^{\times}\underline{v}_2\underline{v}_2^T \\
& + C_1S_2\underline{v}_2^{\times} - (C_1S_2 - S_2)\underline{v}_1\underline{v}_1^T\underline{v}_2^{\times} + S_1S_2\underline{v}_1^{\times}\underline{v}_2^{\times} - \cancel{C_1C_2 I_3} - (C_2 - \cancel{C_1C_2})\underline{v}_1\underline{v}_1^T + S_1C_2\underline{v}_1^{\times} \\
& - (C_1 - \cancel{C_1C_2})\underline{v}_2\underline{v}_2^T - (1-C_1)(1-C_2)\underline{v}_2\underline{v}_2^T\underline{v}_1\underline{v}_1^T - (S_1C_2 - S_1)\underline{v}_2\underline{v}_2^T\underline{v}_1^{\times} + C_1S_2\underline{v}_2^{\times} \\
& - (C_1S_2 - S_2)\underline{v}_2^{\times}\underline{v}_1\underline{v}_1^T - S_1S_2\underline{v}_2^{\times}\underline{v}_1^{\times}
\end{aligned}$$

$$\begin{aligned}
2\sin\theta_3\,\underline{v}_3^{\times} = {} & 2S_1C_2\underline{v}_1^{\times} + (1-C_1)(1-C_2)\left[\underline{v}_1\underline{v}_1^T\underline{v}_2\underline{v}_2^T - \underline{v}_2\underline{v}_2^T\underline{v}_1\underline{v}_1^T\right] + (S_1C_2 - S_1)\left[-\underline{v}_1^{\times}\underline{v}_2\underline{v}_2^T - \underline{v}_2\underline{v}_2^T\underline{v}_1^{\times}\right] \\
& + 2C_1S_2\underline{v}_2^{\times} + (C_1S_2 - S_2)\left[-\underline{v}_1\underline{v}_1^T\underline{v}_2^{\times} - \underline{v}_2^{\times}\underline{v}_1\underline{v}_1^T\right] + S_1S_2\left[\underline{v}_1^{\times}\underline{v}_2^{\times} - \underline{v}_2^{\times}\underline{v}_1^{\times}\right]
\end{aligned}$$

↳ Notice that:

1) If $\underline{w} = \underline{a} \times \underline{b}$, then $\underline{w}^{\times} = \underline{b}\,\underline{a}^T - \underline{a}\,\underline{b}^T$. Thus, notice that:

$$\begin{aligned}
-\underline{v}_1^{\times}\underline{v}_2\underline{v}_2^T - \underline{v}_2\underline{v}_2^T\underline{v}_1^{\times} &= -(\underline{v}_1 \times \underline{v}_2)\underline{v}_2^T + \underline{v}_2\underline{v}_2^T\underline{v}_1^{\times T} = -(\underline{v}_1 \times \underline{v}_2)\underline{v}_2^T + \underline{v}_2(\underline{v}_1^{\times}\underline{v}_2)^T \\
&= \underbrace{\underline{v}_2}_{\underline{b}}\underbrace{(\underline{v}_1 \times \underline{v}_2)^T}_{\underline{a}} - (\underline{v}_1 \times \underline{v}_2)\underline{v}_2^T = [(\underline{v}_1 \times \underline{v}_2) \times \underline{v}_2]^{\times} = -[\underline{v}_2 \times (\underline{v}_1 \times \underline{v}_2)]^{\times}
\end{aligned}$$

Furthermore, it is known that $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a}\cdot\underline{c})\,\underline{b} - (\underline{a}\cdot\underline{b})\,\underline{c}$. Thus:

$$\underline{v}_2 \times (\underline{v}_1 \times \underline{v}_2) = \underbrace{(\underline{v}_2 \cdot \underline{v}_2)}_{=1}\underline{v}_1 - \underbrace{(\underline{v}_2 \cdot \underline{v}_1)}_{\cos\Upsilon}\underline{v}_2 = \underline{v}_1 - \cos\Upsilon\,\underline{v}_2$$

Finally: $-\underline{v}_1^{\times}\underline{v}_2\underline{v}_2^T - \underline{v}_2\underline{v}_2^T\underline{v}_1^{\times} = -[\underline{v}_1 - \cos\Upsilon\,\underline{v}_2]^{\times} = -\underline{v}_1^{\times} + \cos\Upsilon\,\underline{v}_2^{\times}$

2) Using the same properties:

$$\begin{aligned}
-\underline{v}_1\underline{v}_1^T\underline{v}_2^{\times} - \underline{v}_2^{\times}\underline{v}_1\underline{v}_1^T &= \underline{v}_1\underline{v}_1^T\underline{v}_2^{\times T} - (\underline{v}_2 \times \underline{v}_1)\underline{v}_1^T = \underline{v}_1(\underline{v}_2^{\times}\underline{v}_1)^T - (\underline{v}_2 \times \underline{v}_1)\underline{v}_1^T \\
&= \underbrace{\underline{v}_1}_{\underline{b}}\underbrace{(\underline{v}_2 \times \underline{v}_1)^T}_{\underline{a}} - (\underline{v}_2 \times \underline{v}_1)\underline{v}_1^T = [(\underline{v}_2 \times \underline{v}_1) \times \underline{v}_1]^{\times} = -[\underline{v}_1 \times (\underline{v}_2 \times \underline{v}_1)]^{\times}
\end{aligned}$$

$$\Rightarrow \underline{v}_1 \times (\underline{v}_2 \times \underline{v}_1) = \underbrace{(\underline{v}_1 \cdot \underline{v}_1)}_{=1}\underline{v}_2 - \underbrace{(\underline{v}_1 \cdot \underline{v}_2)}_{\cos\Upsilon}\underline{v}_1 = \underline{v}_2 - \cos\Upsilon\,\underline{v}_1$$

Finally: $-\underline{v}_1\underline{v}_1^T\underline{v}_2^{\times} - \underline{v}_2^{\times}\underline{v}_1\underline{v}_1^T = -[\underline{v}_2 - \cos\Upsilon\,\underline{v}_1]^{\times} = -\underline{v}_2^{\times} + \cos\Upsilon\,\underline{v}_1^{\times}$

$$2\sin\theta_3\, \underline{v}_3^\times = 2 S_1 C_2\, \underline{v}_1^\times + (1-C_1)(1-C_2)\left[\underline{v}_1\underline{v}_1^T\underline{v}_2\underline{v}_2^T - \underline{v}_2\underline{v}_2^T\underline{v}_1\underline{v}_1^T\right] + (S_1C_2 - S_1)\left[-\underline{v}_1^\times + \cos\gamma\, \underline{v}_2^\times\right]$$
$$+ 2 C_1 S_2\, \underline{v}_2^\times + (C_1S_2 - S_2)\left[-\underline{v}_2^\times + \cos\gamma\, \underline{v}_1^\times\right] + S_1S_2\left[\underline{v}_1^\times\underline{v}_2^\times - \underline{v}_2^\times\underline{v}_1^\times\right]$$
$$= \left[S_1C_2 + S_1 + S_2(C_1-1)\cos\gamma\right]\underline{v}_1^\times + \left[C_1S_2 + S_2 + S_1(C_2-1)\cos\gamma\right]\underline{v}_2^\times$$
$$+ (1-C_1)(1-C_2)\left[\underline{v}_1\underline{v}_1^T\underline{v}_2\underline{v}_2^T - \underline{v}_2\underline{v}_2^T\underline{v}_1\underline{v}_1^T\right] + S_1S_2\left[\underline{v}_1^\times\underline{v}_2^\times - \underline{v}_2^\times\underline{v}_1^\times\right]$$

$\hookrightarrow$ Notice that:

$$\underline{v}_1\underbrace{\underline{v}_1^T\underline{v}_2}_{\cos\gamma}\underline{v}_2^T - \underline{v}_2\underbrace{\underline{v}_2^T\underline{v}_1}_{\cos\gamma}\underline{v}_1^T = \underbrace{\left[\underline{v}_1\underline{v}_2^T - \underline{v}_2\underline{v}_1^T\right]}_{[\underline{v}_2\times\underline{v}_1]^\times = [\underline{v}_2^\times\underline{v}_1]^\times = -[\underline{v}_1^\times\underline{v}_2]^\times}\cos\gamma = -\left[\underline{v}_1^\times\underline{v}_2\right]^\times\cos\gamma$$

Using the property $a^\times b^\times = b\,a^T - b^T a\, I_3$

$$\underline{v}_1^\times\underline{v}_2^\times - \underline{v}_2^\times\underline{v}_1^\times = \underline{v}_2\underline{v}_1^T - \underline{v}_2^T\underline{v}_1 I_3 - \left(\underline{v}_1\underline{v}_2^T - \underline{v}_1^T\underline{v}_2 I_3\right)$$
$$= \underline{v}_2\underline{v}_1^T - \underline{v}_1\underline{v}_2^T - \cancel{\underline{v}_2^T\underline{v}_1 I_3} + \cancel{\underline{v}_1^T\underline{v}_2 I_3}$$
$$= \underbrace{\underline{v}_2}_{b}\underbrace{\underline{v}_1^T}_{a} - \underline{v}_1\underline{v}_2^T = [\underline{v}_1\times\underline{v}_2]^\times = [\underline{v}_1^\times\underline{v}_2]^\times$$
$$\Rightarrow \underline{w} = \underline{a}\times\underline{b},\quad \underline{w}^\times = \underline{b}\,\underline{a}^T - \underline{a}\,\underline{b}^T$$

$$2\sin\theta_3\,\underline{v}_3^\times = \left[S_1(1+C_2) - S_2(1-C_1)\cos\gamma\right]\underline{v}_1^\times$$
$$+ \left[S_2(1+C_1) - S_1(1-C_2)\cos\gamma\right]\underline{v}_2^\times$$
$$+ \left[S_1S_2 - (1-C_1)(1-C_2)\cos\gamma\right]\left[\underline{v}_1^\times\underline{v}_2\right]^\times$$

Since both sides are matrices written in the same form $([\cdot]^\times)$, then we can compare each element to obtain the following vector equation:

$$2\sin\theta_3\,\underline{v}_3 = +\left[\sin\theta_1(1+\cos\theta_2) - \sin\theta_2(1-\cos\theta_1)\cos\gamma\right]\underline{v}_1$$
$$+ \left[\sin\theta_2(1+\cos\theta_1) - \sin\theta_1(1-\cos\theta_2)\cos\gamma\right]\underline{v}_2$$
$$+ \left[\sin\theta_1\sin\theta_2 - (1-\cos\theta_1)(1-\cos\theta_2)\cos\gamma\right]\underline{v}_1^\times\underline{v}_2$$

This equation can be used to compute the Euler axis of the composed rotation. However, a cleaner expression can be obtained using the half-angles as seen in the following. First, remember that:

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 \qquad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

Thus:

1) $\sin\theta_1(1+\cos\theta_2) - \sin\theta_2(1-\cos\theta_1)\cos\gamma =$

$$2\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\left[\cancel{1} + 2\cos^2\frac{\theta_2}{2} \cancel{-1}\right] - 2\sin\frac{\theta_2}{2}\cos\frac{\theta_2}{2}\left[1 - 2\cos^2\frac{\theta_1}{2} + 1\right]\cos\gamma =$$

$$4\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} - 4\sin\frac{\theta_2}{2}\cos\frac{\theta_2}{2}\underbrace{\left[1 - \cos^2\frac{\theta_1}{2}\right]}_{\sin^2\frac{\theta_1}{2}}\cos\gamma =$$

$$4\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - 4\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\gamma =$$

$$4\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\underbrace{\left[\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\gamma\right]}_{\cos\frac{\theta_3}{2}} = 4\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_3}{2}$$

2) $\sin\theta_2\,(1+\cos\theta_1) - \sin\theta_1\,(1-\cos\theta_2)\cos\gamma =$

$$2\sin\frac{\theta_2}{2}\cos\frac{\theta_2}{2}\left[\cancel{1}+2\cos^2\frac{\theta_1}{2}\cancel{-1}\right] - 2\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\left[1-2\cos^2\frac{\theta_2}{2}+1\right]\cos\gamma =$$

$$4\sin\frac{\theta_2}{2}\cos\frac{\theta_2}{2}\cos^2\frac{\theta_1}{2} - 4\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\underbrace{\left[1-\cos^2\frac{\theta_2}{2}\right]}_{\sin^2\frac{\theta_2}{2}}\cos\gamma =$$

$$4\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - 4\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\gamma =$$

$$4\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\underbrace{\left[\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\gamma\right]}_{\cos\frac{\theta_3}{2}} = 4\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_3}{2}$$

3) $\sin\theta_1\sin\theta_2 - (1-\cos\theta_1)(1-\cos\theta_2)\cos\gamma =$

$$2\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\,2\sin\frac{\theta_2}{2}\cos\frac{\theta_2}{2} - \left[1-2\cos^2\frac{\theta_1}{2}+1\right]\left[1-2\cos^2\frac{\theta_2}{2}+1\right]\cos\gamma =$$

$$4\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - 4\underbrace{\left[1-\cos^2\frac{\theta_1}{2}\right]}_{\sin^2\frac{\theta_1}{2}}\underbrace{\left[1-\cos^2\frac{\theta_2}{2}\right]}_{\sin^2\frac{\theta_2}{2}}\cos\gamma =$$

$$4\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - 4\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\gamma =$$

$$4\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\underbrace{\left[\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\gamma\right]}_{\cos\frac{\theta_3}{2}} = 4\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_3}{2}$$

4) $2\sin\theta_3\,\underline{v}_3 = 2.2\sin\frac{\theta_3}{2}\cos\frac{\theta_3}{2}\,\underline{v}_3 = 4\sin\frac{\theta_3}{2}\cos\frac{\theta_3}{2}\,\underline{v}_3$

Finally:

$$\cancel{4}\sin\frac{\theta_3}{2}\cancel{\cos\frac{\theta_3}{2}}\,\underline{v}_3 = \cancel{4}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cancel{\cos\frac{\theta_3}{2}}\,\underline{v}_1 + \cancel{4}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cancel{\cos\frac{\theta_3}{2}}\,\underline{v}_2 + \cancel{4}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cancel{\cos\frac{\theta_3}{2}}\,\underline{v}_1^{\times}\underline{v}_2$$

$$\underline{v}_3 = \left[\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\,\underline{v}_1 + \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\,\underline{v}_2 + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\,\underline{v}_1^{\times}\underline{v}_2\right]\cdot\frac{1}{\sin\frac{\theta_3}{2}}, \quad \text{if } \sin\theta_3 \neq 0.$$

If $\sin\theta_3 = 0$, then there is two possibilities:

1. $\theta = 0°$: in this case, there is no rotation and the Euler axis is undefined.
2. $\theta = 180°$: in this case, the Euler axis can be obtained by $\underline{C}_{ca} = \underline{v}_3\underline{v}_3^T - \underline{I}_3$.

This case will not be proved here. See problem 2.10 of Hughes's book.