Euler angles

Euler also proved that any direction cosine matrix can be written as a composition of rotations about the reference system axes. For example, it is possible to find ψ, θ, and ϕ in which:

$$\underline{C}_{ba} = \underline{C}_3(\psi)\,\underline{C}_2(\theta)\,\underline{C}_1(\phi)$$

where:

$$\underline{C}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \qquad \underline{C}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \qquad \underline{C}_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The following rotation sequences are valid:

1-2-1	2-1-2	3-1-2
1-2-3	2-1-3	3-1-3
1-3-2	2-3-1	3-2-1

Notice that a DCM can be written in several forms, e.g.:

$$\underline{C}_{ba} = \underline{C}_3(\theta_3)\,\underline{C}_2(\theta_2)\,\underline{C}_1(\theta_1) = \underline{C}_2(\phi_3)\,\underline{C}_1(\phi_2)\,\underline{C}_2(\phi_1)$$

Hence, the Euler angles must be specified together with the rotation axis.

This representation has singularities. There are cases in which a DCM cannot be represented using a specific rotation sequence. For example, let's analyze the case of the sequence 3-1-3 with $\theta_2 = 0$:

$$\underline{C}_{ba} = \begin{bmatrix} \cos(\theta_1+\theta_3) & \sin(\theta_1+\theta_3) & 0 \\ -\sin(\theta_1+\theta_3) & \cos(\theta_1+\theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{C}_3(\theta_3)\,\underline{C}_1(0)\,\underline{C}_3(\theta_1)$$

In this case, we have only 1 rotation of $(\theta_1+\theta_3)^\circ$ about the z-axis. This means that it is impossible to distinguish between θ_1 and θ_3.

↳ For more information, see Gimbal lock.

Given those problems, the Euler angles are rarely used to propagate the attitude. However, they are often used to visualize the attitude since they provide a geometric representation that can be easily interpreted.

Notes:
1) The composition of the rotations is not commutative. Hence, the order matters!
2) $\underline{C}_{ba} = \underline{C}_3(\psi)\,\underline{C}_2(\theta)\,\underline{C}_1(\phi) \Rightarrow \underline{C}_{ba}^T = [\underline{C}_3(\psi)\,\underline{C}_2(\theta)\,\underline{C}_1(\phi)]^T$

$$\underline{C}_{ba}^T = \underline{C}_1^T(\phi)\,\underline{C}_2^T(\theta)\,\underline{C}_3^T(\psi)$$

$$\underline{C}_{ba}^T = \underline{C}_1(-\phi)\,\underline{C}_2(-\theta)\,\underline{C}_3(-\psi)$$

$$\Rightarrow \underline{C}_{ab} = \underline{C}_1(-\phi)\,\underline{C}_2(-\theta)\,\underline{C}_3(-\psi)$$