Euler symmetric parameters

We verified that the Euler angle-axis representation and the Euler angles representation have singularities. Now, we will see a representation that has no singularity, but requires 4 parameters.

Let $\mathcal{F}_a$ and $\mathcal{F}_b$ be two coordinate systems in which the Euler angle-axis is $(\underline{v}, \theta)$. Thus, the Euler symmetric parameters (also called Rodrigues symmetric parameters) are defined as follows:

$$\underline{\varepsilon} = \underline{v} \sin \frac{\theta}{2} \longrightarrow \text{Gibbs / Rodrigues vector}$$

$$\eta = \cos \frac{\theta}{2}$$

Notice that all information in the Euler axis-angle is embedded in these two parameters. Furthermore, we have the following property:

$$\underline{\varepsilon}^T \underline{\varepsilon} + \eta^2 = \underbrace{\underline{v}^T \underline{v}}_{=1} \sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} = \sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} = 1$$

$$\underline{\varepsilon}^T \underline{\varepsilon} + \eta^2 = 1$$

These parameters can be related to the DCM as follows:

$$\underline{C}_{ba} = \cos\theta \, \underline{I}_3 + (1 - \cos\theta)\, \underline{v}\,\underline{v}^T - \sin\theta \, \underline{v}^\times$$

$$= \left[2\cos^2 \frac{\theta}{2} - 1\right] \underline{I}_3 + \left[1 + 1 - 2\cos^2 \frac{\theta}{2}\right] \underline{v}\,\underline{v}^T - 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \underline{v}^\times \longrightarrow \cos\theta = 2\cos^2\frac{\theta}{2} - 1 \quad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= \left[2\cos^2 \frac{\theta}{2} - 1\right] \underline{I}_3 + 2 \underbrace{\left[1 - \cos^2 \frac{\theta}{2}\right]}_{\sin^2 \frac{\theta}{2}} \underline{v}\,\underline{v}^T - 2\cos\frac{\theta}{2} \underbrace{\sin\frac{\theta}{2}\, \underline{v}^\times}_{\left[\sin\frac{\theta}{2}\underline{v}\right]^\times}$$

$$2\eta^2 - 1 = 2\eta^2 - \underline{\varepsilon}^T\underline{\varepsilon} - \eta^2 = \eta^2 - \underline{\varepsilon}^T\underline{\varepsilon}$$

$$= \overbrace{\left[2\cos^2 \frac{\theta}{2} - 1\right]} \underline{I}_3 + 2 \sin^2 \frac{\theta}{2} \underline{v}\,\underline{v}^T - 2\cos\frac{\theta}{2} \left[\sin \frac{\theta}{2} \underline{v}\right]^\times$$

$$\Rightarrow \underline{C}_{ba} = (\eta^2 - \underline{\varepsilon}^T \underline{\varepsilon}) \underline{I}_3 + 2 \underline{\varepsilon}\,\underline{\varepsilon}^T - 2\eta\, \underline{\varepsilon}^\times$$

Composition of rotations

Let $(\underline{\varepsilon}_1, \eta_1)$ be the Euler symmetric parameters that describe the rotation from $\mathcal{F}_a$ to $\mathcal{F}_b$. Let $(\underline{\varepsilon}_2, \eta_2)$ be the Euler symmetric parameters that describe the rotation from $\mathcal{F}_b$ to $\mathcal{F}_c$. We want to find the Euler symmetric parameters that describe the rotation from $\mathcal{F}_a$ to $\mathcal{F}_c$ in terms of $\underline{\varepsilon}_1$, η_1, $\underline{\varepsilon}_2$, and η_2.

From the Euler angle-axis theory, if $\sin\theta_3 \neq 0$, then:

$$\cos\frac{\theta_3}{2} = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\underline{v}_1^T\underline{v}_2$$

$$\underline{v}_3 \sin\frac{\theta_3}{2} = \underline{v}_1 \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \underline{v}_2 \sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + \underline{v}_1^\times \underline{v}_2 \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}$$

Thus:

$$\cos\frac{\theta_3}{2} = \eta_3 = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \left[\mathbf{v}_1 \sin\frac{\theta_1}{2}\right]^T \left[\mathbf{v}_2 \sin\frac{\theta_2}{2}\right]$$

$$\Rightarrow \eta_3 = \eta_1 \eta_2 - \varepsilon_1^T \varepsilon_2$$

$$\mathbf{v}_3 \sin\frac{\theta_3}{2} = \varepsilon_3 = \left[\mathbf{v}_1 \sin\frac{\theta_1}{2}\right]\cos\frac{\theta_2}{2} + \left[\mathbf{v}_2 \sin\frac{\theta_2}{2}\right]\cos\frac{\theta_1}{2} + \left[\mathbf{v}_1 \sin\frac{\theta_1}{2}\right]^\times \left[\mathbf{v}_2 \sin\frac{\theta_2}{2}\right]$$

$$\Rightarrow \varepsilon_3 = \eta_2 \varepsilon_1 + \eta_1 \varepsilon_2 + \varepsilon_1^\times \varepsilon_2$$

If $\theta_3 = 180°$, then

$$\eta_3 = 0 \quad \text{and} \quad \varepsilon_3 = \mathbf{v}_3 \longrightarrow \eta_1 \eta_2 = \varepsilon_1^T \varepsilon_2 = \varepsilon_2^T \varepsilon_1$$

Thus:

$$C_{ca} = 2\mathbf{v}_3 \mathbf{v}_3^T - I_3 = 2\varepsilon_3 \varepsilon_3^T - I_3$$

Suppose that we can also write $\varepsilon_3 = \eta_2 \varepsilon_1 + \eta_1 \varepsilon_2 + \varepsilon_1^\times \varepsilon_2$. In this case:

$$C_{ca} = 2\left[\eta_2 \varepsilon_1 + \eta_1 \varepsilon_2 + \varepsilon_1^\times \varepsilon_2\right]\left[\eta_2 \varepsilon_1 + \eta_1 \varepsilon_2 + \varepsilon_1^\times \varepsilon_2\right]^T - I_3$$

$$= 2\left[\eta_2 \varepsilon_1 + \eta_1 \varepsilon_2 + \varepsilon_1^\times \varepsilon_2\right]\left[\eta_2 \varepsilon_1^T + \eta_1 \varepsilon_2^T - \varepsilon_2^T \varepsilon_1^\times\right] - I_3$$

$$= 2\Big[\eta_2^2 \varepsilon_1 \varepsilon_1^T + \overbrace{\eta_1 \eta_2}^{\varepsilon_1^T \varepsilon_2} \varepsilon_1 \varepsilon_2^T - \eta_2 \varepsilon_1 \varepsilon_2^T \varepsilon_1^\times + \overbrace{\eta_1 \eta_2}^{\varepsilon_2^T \varepsilon_1} \varepsilon_2 \varepsilon_1^T + \eta_1^2 \varepsilon_2 \varepsilon_2^T - \eta_1 \varepsilon_2 \varepsilon_2^T \varepsilon_1^\times$$
$$+ \eta_2 \varepsilon_1^\times \varepsilon_2 \varepsilon_1^T + \eta_1 \varepsilon_1^\times \varepsilon_2 \varepsilon_2^T - \varepsilon_1^\times \varepsilon_2 \varepsilon_2^T \varepsilon_1^\times\Big] - I_3$$

$$= 2\Big[\eta_2^2 \varepsilon_1 \varepsilon_1^T + \varepsilon_1 \varepsilon_1^T \varepsilon_2 \varepsilon_2^T - \eta_2 \varepsilon_1 \varepsilon_2^T \varepsilon_1^\times + \varepsilon_2 \varepsilon_2^T \varepsilon_1 \varepsilon_1^T + \eta_1^2 \varepsilon_2 \varepsilon_2^T - \eta_1 \varepsilon_2 \varepsilon_2^T \varepsilon_1^\times$$
$$+ \eta_2 \varepsilon_1^\times \varepsilon_2 \varepsilon_1^T + \eta_1 \varepsilon_1^\times \varepsilon_2 \varepsilon_2^T - \varepsilon_1^\times \varepsilon_2 \varepsilon_2^T \varepsilon_1^\times + \eta_1 \varepsilon_2 \varepsilon_2^T \varepsilon_1^\times - \eta_1 \varepsilon_2 \varepsilon_2^T \varepsilon_1^\times\Big] - I_3$$

$\hookrightarrow$ Using that, $\mathbf{M}\mathbf{u}^\times + \mathbf{u}^\times \mathbf{M}^T = \mathrm{tr}[\mathbf{M}]\mathbf{u}^\times - [\mathbf{M}^T \mathbf{u}]^\times$, $\mathbf{u} \in \mathbb{R}^3$, $\mathbf{M} \in \mathbb{R}^{3\times3}$, then:

$$\eta_1\Big[\overset{\mathbf{u}}{\varepsilon_1^\times}\overbrace{\varepsilon_2 \varepsilon_2^T}^{\mathbf{M}^T} + \overbrace{\varepsilon_2 \varepsilon_2^T}^{\mathbf{M}}\overset{\mathbf{u}}{\varepsilon_1^\times}\Big] = \eta_1\Big[\mathrm{tr}(\varepsilon_2 \varepsilon_2^T)\varepsilon_1^\times - \big[\varepsilon_2 \underbrace{\varepsilon_2^T \varepsilon_1}_{\text{scalar}}\big]^\times\Big]$$

Since $\mathrm{tr}[\varepsilon_2 \varepsilon_2^T] = \mathrm{tr}\begin{bmatrix} \varepsilon_{2,x}^2 & \cdot & \cdot \\ \cdot & \varepsilon_{2,y}^2 & \cdot \\ \cdot & \cdot & \varepsilon_{2,z}^2 \end{bmatrix} = \varepsilon_{2,x}^2 + \varepsilon_{2,y}^2 + \varepsilon_{2,z}^2 = \varepsilon_2^T \varepsilon_2$, then:

$$\eta_1\left[\varepsilon_1^\times \varepsilon_2 \varepsilon_2^T + \varepsilon_2 \varepsilon_2^T \varepsilon_1^\times\right] = \eta_1\left[\varepsilon_2^T \varepsilon_2 \varepsilon_1^\times - \varepsilon_2^T \varepsilon_1 \varepsilon_2^\times\right]$$

$$\underline{C}_{ca} = 2\left[\eta_2^2\, \underline{\varepsilon}_1 \underline{\varepsilon}_1^T + \underline{\varepsilon}_1 \underline{\varepsilon}_1^T \underline{\varepsilon}_2 \underline{\varepsilon}_2^T - \eta_2\, \underline{\varepsilon}_1 \underline{\varepsilon}_2^T \underline{\varepsilon}_1^\times + \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1 \underline{\varepsilon}_1^T + \eta_1^2\, \underline{\varepsilon}_2 \underline{\varepsilon}_2^T - 2\eta_1\, \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1^\times \right.$$

$$\left. + \eta_2\, \underline{\varepsilon}_1^\times \underline{\varepsilon}_2 \underline{\varepsilon}_1^T + \eta_1 \underline{\varepsilon}_2^T \underline{\varepsilon}_2\, \underline{\varepsilon}_1^\times - \eta_1\, \underline{\varepsilon}_2^T \underline{\varepsilon}_1 \underline{\varepsilon}_2^\times - \underline{\varepsilon}_1^\times \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1^\times - \eta_2 \underline{\varepsilon}_1^\times \underline{\varepsilon}_2 \underline{\varepsilon}_1^T + \eta_2\, \underline{\varepsilon}_1^\times \underline{\varepsilon}_2 \underline{\varepsilon}_1^T \right] - \underline{I}_3$$

↳ Using that: $\underline{M}\,\underline{u}^\times + \underline{u}^\times \underline{M}^T = tr[\underline{M}]\,\underline{u}^\times - [\underline{M}^T \underline{u}]^\times$, $\underline{u} \in \mathbb{R}^3$, $\underline{M} \in \mathbb{R}^{3\times3}$, then:

$$-\eta_2\left[\overbrace{\underline{\varepsilon}_1 \underline{\varepsilon}_2^T}^{\underline{m}}\, \overset{\underline{u}}{\underline{\varepsilon}_1^\times} + \overset{\underline{u}}{\underline{\varepsilon}_1^\times}\, \overbrace{\underline{\varepsilon}_2 \underline{\varepsilon}_1^T}^{\underline{m}^T}\right] = -\eta_2\left[tr[\underline{\varepsilon}_1 \underline{\varepsilon}_2^T]\, \underline{\varepsilon}_1^\times - [\underline{\varepsilon}_2 \underbrace{\underline{\varepsilon}_1^T \underline{\varepsilon}_1}_{\text{scalar}}]^\times\right]$$

Since $tr[\underline{\varepsilon}_1 \underline{\varepsilon}_2^T] = tr\begin{bmatrix} \varepsilon_{1,x}\varepsilon_{2,x} & \cdot & \cdot \\ \cdot & \varepsilon_{1,y}\varepsilon_{2,y} & \cdot \\ \cdot & \cdot & \varepsilon_{1,z}\varepsilon_{2,z} \end{bmatrix} = \varepsilon_{1,x}\varepsilon_{2,x} + \varepsilon_{1,y}\varepsilon_{2,y} + \varepsilon_{1,z}\varepsilon_{2,z} = \underline{\varepsilon}_1^T \underline{\varepsilon}_2$, then:

$$-\eta_2\left[\underline{\varepsilon}_1 \underline{\varepsilon}_2^T \underline{\varepsilon}_1^\times + \underline{\varepsilon}_1^\times \underline{\varepsilon}_2 \underline{\varepsilon}_1^T\right] = -\eta_2\left[\underline{\varepsilon}_1^T \underline{\varepsilon}_2\, \underline{\varepsilon}_1^\times - \underline{\varepsilon}_1^T \underline{\varepsilon}_1 \underline{\varepsilon}_2^\times\right]$$

$$\underline{C}_{ca} = 2\left[\eta_2^2\, \underline{\varepsilon}_1 \underline{\varepsilon}_1^T + \underline{\varepsilon}_1 \underline{\varepsilon}_1^T \underline{\varepsilon}_2 \underline{\varepsilon}_2^T + 2\eta_2\, \underline{\varepsilon}_1^\times \underline{\varepsilon}_2\, \underline{\varepsilon}_1^T + \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1 \underline{\varepsilon}_1^T + \eta_1^2\, \underline{\varepsilon}_2 \underline{\varepsilon}_2^T - 2\eta_1\, \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1^\times \right.$$

$$\left. - \eta_2\, \overbrace{\underline{\varepsilon}_1^T \underline{\varepsilon}_2}^{\eta_1\eta_2}\, \underline{\varepsilon}_1^\times + \eta_2\, \underline{\varepsilon}_1^T \underline{\varepsilon}_1\, \underline{\varepsilon}_2^\times + \eta_1\, \underline{\varepsilon}_2^T \underline{\varepsilon}_2\, \underline{\varepsilon}_1^\times - \eta_1\, \overbrace{\underline{\varepsilon}_2^T \underline{\varepsilon}_1}^{\eta_1\eta_2}\, \underline{\varepsilon}_2^\times - \underline{\varepsilon}_1^\times \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1^\times \right] - \underline{I}_3$$

$$\underline{C}_{ca} = 2\left[\eta_2^2\, \underline{\varepsilon}_1 \underline{\varepsilon}_1^T + \underline{\varepsilon}_1 \underline{\varepsilon}_1^T \underline{\varepsilon}_2 \underline{\varepsilon}_2^T + 2\eta_2\, \underline{\varepsilon}_1^\times \underline{\varepsilon}_2\, \underline{\varepsilon}_1^T + \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1 \underline{\varepsilon}_1^T + \eta_1^2\, \underline{\varepsilon}_2 \underline{\varepsilon}_2^T - 2\eta_1 \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1^\times \right.$$

$$\left. - \eta_1\, (\eta_2^2 - \underline{\varepsilon}_2^T \underline{\varepsilon}_2)\, \underline{\varepsilon}_1^\times - \eta_2\, (\eta_1^2 - \underline{\varepsilon}_1^T \underline{\varepsilon}_1)\, \underline{\varepsilon}_2^\times - \underline{\varepsilon}_1^\times \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1^\times \right] - \underline{I}_3 =$$

↳ Using that $\underline{u}^\times \underline{v}^\times - \underline{v}^\times \underline{u}^\times = \underline{v}\,\underline{u}^T - \underline{u}\,\underline{v}^T$, $\underline{u}, \underline{v} \in \mathbb{R}^3$, then:

$$\underline{\varepsilon}_1 \underline{\varepsilon}_1^T \underline{\varepsilon}_2 \underline{\varepsilon}_2^T + \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1 \underline{\varepsilon}_1^T = \underline{\varepsilon}_1^T \underline{\varepsilon}_2\left[\overset{\underline{v}}{\underline{\varepsilon}_1}\, \overset{\underline{u}^T}{\underline{\varepsilon}_2^T} + \overset{\underline{u}}{\underline{\varepsilon}_2}\, \overset{\underline{v}^T}{\underline{\varepsilon}_1^T} - \underline{\varepsilon}_2 \underline{\varepsilon}_1^T + \underline{\varepsilon}_2 \underline{\varepsilon}_1^T\right] =$$

$$\underline{\varepsilon}_1^T \underline{\varepsilon}_2\left[\underline{\varepsilon}_2^\times \underline{\varepsilon}_1^\times - \underline{\varepsilon}_1^\times \underline{\varepsilon}_2^\times + 2\, \underline{\varepsilon}_2 \underline{\varepsilon}_1^T\right] = \underline{\varepsilon}_1^T \underline{\varepsilon}_2\, \underline{\varepsilon}_2^\times \underline{\varepsilon}_1^\times - \underline{\varepsilon}_1^T \underline{\varepsilon}_2\, \underline{\varepsilon}_1^\times \underline{\varepsilon}_2^\times + 2\, \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1 \underline{\varepsilon}_1^T$$

$$\underline{C}_{ca} = 2\left[\eta_2^2\, \underline{\varepsilon}_1 \underline{\varepsilon}_1^T + 2\, \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1 \underline{\varepsilon}_1^T + \underline{\varepsilon}_1^T \underline{\varepsilon}_2\, \underline{\varepsilon}_2^\times \underline{\varepsilon}_1^\times - \underline{\varepsilon}_1^T \underline{\varepsilon}_2\, \underline{\varepsilon}_1^\times \underline{\varepsilon}_2^\times + 2\eta_2\, \underline{\varepsilon}_1^\times \underline{\varepsilon}_2\, \underline{\varepsilon}_1^T + \eta_1^2\, \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \right.$$

$$\left. - 2\eta_1 \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1^\times - \eta_1 (\eta_2^2 - \underline{\varepsilon}_2^T \underline{\varepsilon}_2)\, \underline{\varepsilon}_1^\times - \eta_2\, (\eta_1^2 - \underline{\varepsilon}_1^T \underline{\varepsilon}_1)\, \underline{\varepsilon}_2^\times - \underline{\varepsilon}_1^\times \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1^\times \right] - \underline{I}_3$$

↳ Using that $\underline{u}^\times \underline{v}^\times = \underline{v}\,\underline{u}^T - \underline{v}^T \underline{u}\, \underline{I}_3$, then

$$-\underline{\varepsilon}_1^T \underline{\varepsilon}_2\, \overset{\underline{u}^\times}{\underline{\varepsilon}_1^\times}\, \overset{\underline{v}^\times}{\underline{\varepsilon}_2^\times} = -\overbrace{\underline{\varepsilon}_1^T \underline{\varepsilon}_2}^{\eta_1\eta_2}\left[\underline{\varepsilon}_2 \underline{\varepsilon}_1^T - \overbrace{\underline{\varepsilon}_2^T \underline{\varepsilon}_1}^{\eta_1\eta_2}\, \underline{I}_3\right] = -\underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1 \underline{\varepsilon}_1^T + \eta_1^2\eta_2^2\, \underline{I}_3$$

$$\underline{C}_{ca} = 2\left[\eta_1^2\eta_2^2\, \underline{I}_3 + \eta_2^2 \underline{\varepsilon}_1 \underline{\varepsilon}_1^T + 2\, \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1 \underline{\varepsilon}_1^T + \overbrace{\underline{\varepsilon}_1^T \underline{\varepsilon}_2}^{\eta_1\eta_2}\, \underline{\varepsilon}_2^\times \underline{\varepsilon}_1^\times + \eta_1^2\, \underline{\varepsilon}_2 \underline{\varepsilon}_2^T + 2\eta_2\, \overbrace{\underline{\varepsilon}_1^\times \underline{\varepsilon}_2}^{-\underline{\varepsilon}_2^\times \underline{\varepsilon}_1}\, \underline{\varepsilon}_1^T \right.$$

$$\left. - 2\eta_1 \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1^\times - \eta_1\, (\eta_2^2 - \underline{\varepsilon}_2^T \underline{\varepsilon}_2)\, \underline{\varepsilon}_1^\times - \eta_2\, (\eta_1^2 - \underline{\varepsilon}_1^T \underline{\varepsilon}_1)\, \underline{\varepsilon}_2^\times - \underline{\varepsilon}_1^\times \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1^\times - \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1 \underline{\varepsilon}_1^T \right] - \underline{I}_3$$

$$\underline{C}_{ca} = 2\eta_1^2\eta_2^2\, \underline{I}_3 + 2\eta_2^2\, \underline{\varepsilon}_1 \underline{\varepsilon}_1^T + 4\, \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1 \underline{\varepsilon}_1^T + 2\eta_1\eta_2\, \underline{\varepsilon}_2^\times \underline{\varepsilon}_1^\times + 2\eta_1^2\, \underline{\varepsilon}_2 \underline{\varepsilon}_2^T - 4\eta_2\, \underline{\varepsilon}_2^\times \underline{\varepsilon}_1 \underline{\varepsilon}_1^T$$

$$-4\eta_1\, \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1^\times - 2\eta_1\, (\eta_2^2 - \underline{\varepsilon}_2^T \underline{\varepsilon}_2)\, \underline{\varepsilon}_1^\times - 2\eta_2\, (\eta_1^2 - \underline{\varepsilon}_1^T \underline{\varepsilon}_1)\, \underline{\varepsilon}_2^\times - 2\underline{\varepsilon}_1^\times \underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1^\times$$

$$-2\underline{\varepsilon}_2 \underline{\varepsilon}_2^T \underline{\varepsilon}_1 \underline{\varepsilon}_1^T - \underline{I}_3$$

Notice that:

$$-2\,\epsilon_1^\times \epsilon_2 \epsilon_2^T \epsilon_1^\times = -2\,\epsilon_1^\times\left[\overbrace{\epsilon_2\epsilon_2^T - \epsilon_2^T\epsilon_2 I_3}^{\text{Using that } u^\times v^\times = v u^T - v^T u\, I_3} + \epsilon_2^T\epsilon_2 I_3\right]\epsilon_1^\times$$

$$= -2\,\epsilon_1^\times\left[\epsilon_2^\times\epsilon_2^\times + \epsilon_2^T\epsilon_2 I_3\right]\epsilon_1^\times$$

$$= -2\,\epsilon_1^\times\epsilon_2^\times\epsilon_2^\times\epsilon_1^\times - 2\epsilon_2^T\epsilon_2\,\epsilon_1^\times\epsilon_1^\times \longrightarrow u^\times v^\times = v u^T - u^T v\, I_3$$

$$= -2\left[\epsilon_2\epsilon_1^T - \epsilon_2^T\epsilon_1 I\right]\left[\epsilon_1\epsilon_2^T - \epsilon_1^T\epsilon_2 I_3\right] - 2\epsilon_2^T\epsilon_2\,\epsilon_1^\times\epsilon_1^\times$$

$$= -2\,\epsilon_2\underbrace{\epsilon_1^T\epsilon_1}_{\text{scalar}}\epsilon_2^T + 2\epsilon_2\epsilon_1^T\underbrace{\epsilon_1^T\epsilon_2}_{\text{scalar}} + 2\underbrace{\epsilon_2^T\epsilon_1}_{\text{Scalar}}\epsilon_1\epsilon_2^T - 2\underbrace{\epsilon_2^T\epsilon_1}_{\eta_1\eta_2}\underbrace{\epsilon_1^T\epsilon_2}_{\eta_1\eta_2} I_3$$
$$- 2\epsilon_2^T\epsilon_2\,\epsilon_1^\times\epsilon_1^\times$$

$$= -2\epsilon_1^T\epsilon_1\,\epsilon_2\epsilon_2^T + 2\epsilon_2\,\epsilon_1^T\epsilon_2\,\epsilon_1^T + 2\epsilon_1\,\epsilon_2^T\epsilon_1\,\epsilon_2^T - 2\eta_1^2\eta_2^2 I_3$$
$$- 2\epsilon_2^T\epsilon_2\,\epsilon_1^\times\epsilon_1^\times$$

$$= -2\epsilon_1^T\epsilon_1\,\epsilon_2\epsilon_2^T + 2\epsilon_2\,\epsilon_2^T\epsilon_1\,\epsilon_1^T + 2\epsilon_1\,\epsilon_2^T\epsilon_1\,\epsilon_2^T - 2\eta_1^2\eta_2^2 I_3$$
$$- 2\epsilon_2^T\epsilon_2\,\epsilon_1^\times\epsilon_1^\times$$

$$C_{ca} = 2\cancel{\eta_1^2\eta_2^2 I_3} + 2\eta_2^2\epsilon_1\epsilon_1^T + 4\epsilon_2\epsilon_2^T\epsilon_1\epsilon_1^T + 2\eta_1\eta_2\,\epsilon_2^\times\epsilon_1^\times + 2\eta_1^2\epsilon_2\epsilon_2^T - 4\eta_2\,\epsilon_2^\times\epsilon_1\epsilon_1^T$$
$$-4\eta_1\,\epsilon_2\epsilon_2^T\epsilon_1^\times - 2\eta_1(\eta_2^2 - \epsilon_2^T\epsilon_2)\epsilon_1^\times - 2\eta_2(\eta_1^2 - \epsilon_1^T\epsilon_1)\epsilon_2^\times - 2\epsilon_1^T\epsilon_1\,\epsilon_2\epsilon_2^T - 2\cancel{\eta_1^2\eta_2^2 I_3}$$
$$+ 2\epsilon_2\cancel{\epsilon_2^T}\epsilon_1\epsilon_1^T + 2\epsilon_1\epsilon_2^T\epsilon_1\epsilon_2^T - 2\epsilon_2^T\epsilon_2\,\epsilon_1^\times\epsilon_1^\times - 2\epsilon_2\cancel{\epsilon_2^T}\epsilon_1\epsilon_1^T - I_3$$

$$C_{ca} = +2\eta_2^2\epsilon_1\epsilon_1^T + 4\epsilon_2\epsilon_2^T\epsilon_1\epsilon_1^T + 2\eta_1\eta_2\,\epsilon_2^\times\epsilon_1^\times + 2\eta_1^2\epsilon_2\epsilon_2^T - 4\eta_2\,\epsilon_2^\times\epsilon_1\epsilon_1^T$$
$$-4\eta_1\,\epsilon_2\epsilon_2^T\epsilon_1^\times - 2\eta_1(\eta_2^2 - \epsilon_2^T\epsilon_2)\epsilon_1^\times - 2\eta_2(\eta_1^2 - \epsilon_1^T\epsilon_1)\epsilon_2^\times - 2\epsilon_1^T\epsilon_1\,\epsilon_2\epsilon_2^T$$
$$+ 2\epsilon_1\epsilon_2^T\epsilon_1\epsilon_2^T - 2\epsilon_2^T\epsilon_2\,\epsilon_1^\times\epsilon_1^\times - I_3$$

$$C_{ca} = 2\eta_2^2\epsilon_1\epsilon_1^T + 4\epsilon_2\epsilon_2^T\epsilon_1\epsilon_1^T + 2\eta_1\eta_2\,\epsilon_2^\times\epsilon_1^\times + 2(\eta_1^2 - \epsilon_1^T\epsilon_1)\epsilon_2\epsilon_2^T - 4\eta_2\,\epsilon_2^\times\epsilon_1\epsilon_1^T$$
$$-4\eta_1\,\epsilon_2\epsilon_2^T\epsilon_1^\times - 2\eta_1(\eta_2^2 - \epsilon_2^T\epsilon_2)\epsilon_1^\times - 2\eta_2(\eta_1^2 - \epsilon_1^T\epsilon_1)\epsilon_2^\times + 2\epsilon_1\epsilon_2^T\epsilon_1\epsilon_2^T$$
$$-2\epsilon_2^T\epsilon_2\,\epsilon_1^\times\epsilon_1^\times - I_3$$

↳ Using that $u^\times u^\times = u u^T - u^T u\, I_3$

$$-2\epsilon_2^T\epsilon_2\,\epsilon_1^\times\epsilon_1^\times = -2\epsilon_2^T\epsilon_2\left[\epsilon_1\epsilon_1^T - \epsilon_1^T\epsilon_1 I_3\right] = -2\epsilon_2^T\epsilon_2\,\epsilon_1\epsilon_1^T + 2\epsilon_2^T\epsilon_2\,\epsilon_1^T\epsilon_1 I_3$$

$$C_{ca} = 2\eta_2^2\epsilon_1\epsilon_1^T + 4\epsilon_2\epsilon_2^T\epsilon_1\epsilon_1^T + 2\eta_1\eta_2\,\epsilon_2^\times\epsilon_1^\times + 2(\eta_1^2 - \epsilon_1^T\epsilon_1)\epsilon_2\epsilon_2^T - 4\eta_2\,\epsilon_2^\times\epsilon_1\epsilon_1^T$$
$$-4\eta_1\,\epsilon_2\epsilon_2^T\epsilon_1^\times - 2\eta_1(\eta_2^2 - \epsilon_2^T\epsilon_2)\epsilon_1^\times - 2\eta_2(\eta_1^2 - \epsilon_1^T\epsilon_1)\epsilon_2^\times + 2\epsilon_1\epsilon_2^T\epsilon_1\epsilon_2^T$$
$$-2\epsilon_2^T\epsilon_2\,\epsilon_1\epsilon_1^T + 2\epsilon_2^T\epsilon_2\,\epsilon_1^T\epsilon_1 I_3 - I_3$$

$$C_{ca} = 2(\eta_2^2 - \epsilon_2^T\epsilon_2)\epsilon_1\epsilon_1^T + 4\epsilon_2\epsilon_2^T\epsilon_1\epsilon_1^T + 2\eta_1\eta_2\,\epsilon_2^\times\epsilon_1^\times + 2(\eta_1^2 - \epsilon_1^T\epsilon_1)\epsilon_2\epsilon_2^T - 4\eta_2\,\epsilon_2^\times\epsilon_1\epsilon_1^T$$
$$-4\eta_1\,\epsilon_2\epsilon_2^T\epsilon_1^\times - 2\eta_1(\eta_2^2 - \epsilon_2^T\epsilon_2)\epsilon_1^\times - 2\eta_2(\eta_1^2 - \epsilon_1^T\epsilon_1)\epsilon_2^\times + 2\epsilon_1\epsilon_2^T\epsilon_1\epsilon_2^T$$
$$+\left(2\epsilon_2^T\epsilon_2\,\epsilon_1^T\epsilon_1 - 1\right)I_3 + 2\eta_1\eta_2\,\epsilon_2^\times\epsilon_1 - 2\eta_1\eta_2\,\epsilon_2^\times\epsilon_1$$

Notice that: $\quad \underline{u}^\times \underline{v}^\times = \underline{u}\,\underline{v}^T - \underline{u}^T \underline{v}\, \underline{I}_3$

$$-2\eta_1\eta_2\, \underline{\epsilon}_2^\times \underline{\epsilon}_1 = -2\eta_1\eta_2 \big[\underline{\epsilon}_1 \underline{\epsilon}_2^T - \overbrace{\underline{\epsilon}_1^T \underline{\epsilon}_2}^{\eta_1\eta_2}\, \underline{I}_3\big] = -2\eta_1\eta_2\, \underline{\epsilon}_1\underline{\epsilon}_2^T + 2\eta_1^2\eta_2^2\, \underline{I}_3$$

$$\underline{C}_{ca} = 2(\eta_2^2 - \underline{\epsilon}_2^T\underline{\epsilon}_2)\underline{\epsilon}_1\underline{\epsilon}_1^T + 4\,\underline{\epsilon}_2\underline{\epsilon}_2^T\underline{\epsilon}_1\underline{\epsilon}_1^T + 4\eta_1\eta_2\,\underline{\epsilon}_2^\times\underline{\epsilon}_1^\times + 2(\eta_1^2 - \underline{\epsilon}_1^T\underline{\epsilon}_1)\underline{\epsilon}_2\underline{\epsilon}_2^T - 4\eta_2\,\underline{\epsilon}_2^\times\underline{\epsilon}_1\underline{\epsilon}_1^T$$
$$-4\eta_1\,\underline{\epsilon}_2\underline{\epsilon}_2^T\underline{\epsilon}_1^\times - 2\eta_1(\eta_2^2 - \underline{\epsilon}_2^T\underline{\epsilon}_2)\underline{\epsilon}_1^\times - 2\eta_2(\eta_1^2 - \underline{\epsilon}_1^T\underline{\epsilon}_1)\underline{\epsilon}_2^\times + \underbrace{2\,\underline{\epsilon}_1\cancel{\underline{\epsilon}_2^T}\underline{\epsilon}_1\underline{\epsilon}_2^T}$$
$$+\left(2\,\underline{\epsilon}_2^T\underline{\epsilon}_2\,\underline{\epsilon}_1^T\underline{\epsilon}_1 - 1\right)\underline{I}_3 - 2\eta_1\cancel{\eta_2}\,\underline{\epsilon}_1\underline{\epsilon}_2^T + 2\eta_1^2\eta_2^2\,\underline{I}_3$$

$$2\,\underline{\epsilon}_2^T\underline{\epsilon}_1\,\underline{\epsilon}_1\underline{\epsilon}_2^T = 2\eta_1\eta_2\,\underline{\epsilon}_1\underline{\epsilon}_2^T$$

$$\underline{C}_{ca} = 2(\eta_2^2 - \underline{\epsilon}_2^T\underline{\epsilon}_2)\underline{\epsilon}_1\underline{\epsilon}_1^T + 4\,\underline{\epsilon}_2\underline{\epsilon}_2^T\underline{\epsilon}_1\underline{\epsilon}_1^T + 4\eta_1\eta_2\,\underline{\epsilon}_2^\times\underline{\epsilon}_1^\times + 2(\eta_1^2 - \underline{\epsilon}_1^T\underline{\epsilon}_1)\underline{\epsilon}_2\underline{\epsilon}_2^T - 4\eta_2\,\underline{\epsilon}_2^\times\underline{\epsilon}_1\underline{\epsilon}_1^T$$
$$-4\eta_1\,\underline{\epsilon}_2\underline{\epsilon}_2^T\underline{\epsilon}_1^\times - 2\eta_1(\eta_2^2 - \underline{\epsilon}_2^T\underline{\epsilon}_2)\underline{\epsilon}_1^\times - 2\eta_2(\eta_1^2 - \underline{\epsilon}_1^T\underline{\epsilon}_1)\underline{\epsilon}_2^\times +$$
$$+\left(2\eta_1^2\eta_2^2 + 2\,\underline{\epsilon}_2^T\underline{\epsilon}_2\,\underline{\epsilon}_1^T\underline{\epsilon}_1 - 1\right)\underline{I}_3$$

→ Notice that:

$$2\eta_1^2\eta_2^2 + 2\,\underline{\epsilon}_2^T\underline{\epsilon}_2\,\underline{\epsilon}_1^T\underline{\epsilon}_1 - 1 = \eta_1^2\eta_2^2 + \eta_1^2\underbrace{\eta_2^2}_{1-\underline{\epsilon}_2^T\underline{\epsilon}_2} + \overbrace{\underline{\epsilon}_2^T\underline{\epsilon}_2}^{=1-\eta_2^2}\,\underline{\epsilon}_1^T\underline{\epsilon}_1 + \underline{\epsilon}_2^T\underline{\epsilon}_2\,\underline{\epsilon}_1^T\underline{\epsilon}_1 - 1$$
$$\eta_1^2\eta_2^2 + \cancel{\eta_1^2} - \eta_1^2\,\underline{\epsilon}_2^T\underline{\epsilon}_2 + \cancel{\underline{\epsilon}_1^T}\underline{\epsilon}_1 - \underline{\epsilon}_1^T\underline{\epsilon}_1\,\eta_2^2 + \underline{\epsilon}_2^T\underline{\epsilon}_2\,\underline{\epsilon}_1^T\underline{\epsilon}_1\cancel{-1}$$
$$= (\eta_1^2 - \underline{\epsilon}_1^T\underline{\epsilon}_1)(\eta_2^2 - \underline{\epsilon}_2^T\underline{\epsilon}_2)$$

$$\underline{C}_{ca} = (\eta_1^2 - \underline{\epsilon}_1^T\underline{\epsilon}_1)(\eta_2^2 - \underline{\epsilon}_2^T\underline{\epsilon}_2)\,\underline{I}_3 + 2(\eta_2^2 - \underline{\epsilon}_2^T\underline{\epsilon}_2)\underline{\epsilon}_1\underline{\epsilon}_1^T - 2\eta_1(\eta_2^2 - \underline{\epsilon}_2^T\underline{\epsilon}_2)\underline{\epsilon}_1^\times$$
$$+2(\eta_1^2 - \underline{\epsilon}_1^T\underline{\epsilon}_1)\underline{\epsilon}_2\underline{\epsilon}_2^T + 4\,\underline{\epsilon}_2\underline{\epsilon}_2^T\underline{\epsilon}_1\underline{\epsilon}_1^T - 4\eta_1\,\underline{\epsilon}_2\underline{\epsilon}_2^T\underline{\epsilon}_1^\times$$
$$-2(\eta_1^2 - \underline{\epsilon}_1^T\underline{\epsilon}_1)\underline{\epsilon}_2^\times - 4\eta_2\,\underline{\epsilon}_2^\times\underline{\epsilon}_1\underline{\epsilon}_1^T + 4\eta_1\eta_2\,\underline{\epsilon}_2^\times\underline{\epsilon}_1^\times$$

$$\underline{C}_{ca} = \underbrace{\left[(\eta_2^2 - \underline{\epsilon}_2^T\underline{\epsilon}_2)\underline{I}_3 + 2\underline{\epsilon}_2\underline{\epsilon}_2^T - 2\eta_2\,\underline{\epsilon}_2^\times\right]}_{\underline{C}_{cb}}\underbrace{\left[(\eta_1^2 - \underline{\epsilon}_1^T\underline{\epsilon}_1)\underline{I}_3 + 2\underline{\epsilon}_1\underline{\epsilon}_1^T - 2\eta_1\,\underline{\epsilon}_1^\times\right]}_{\underline{C}_{ba}} = \underline{C}_{cb}\,\underline{C}_{ba}$$

Thus, if $\theta_3 = 180°$ and $\underline{\epsilon}_3 = \eta_2\underline{\epsilon}_1 + \eta_1\underline{\epsilon}_2 + \underline{\epsilon}_1^\times\underline{\epsilon}_2$ we obtain the same rotation $\underline{C}_{cb}\,\underline{C}_{ba}$.

Finally, if $\theta_3 = 0°$, then no rotation occurs. In this case, every axis is the Euler axis. Hence, we can select $\underline{\epsilon}_3 = \eta_2\underline{\epsilon}_1 + \eta_1\underline{\epsilon}_2 + \underline{\epsilon}_1^\times\underline{\epsilon}_2$.

Conclusion:

If $(\underline{\epsilon}_1, \eta_1)$ are the Euler symmetric parameters that describe the rotation from $\mathcal{F}_a$ to $\mathcal{F}_b$, and $(\underline{\epsilon}_2, \eta_2)$ are the Euler symmetric parameters that describe the rotation from $\mathcal{F}_b$ to $\mathcal{F}_c$, then the Euler symmetric parameters $(\underline{\epsilon}_3, \eta_3)$ that describes the rotation from $\mathcal{F}_a$ to $\mathcal{F}_c$ can be computed by:

$$\eta_3 = \eta_1\eta_2 - \underline{\epsilon}_1^T\underline{\epsilon}_2$$
$$\underline{\epsilon}_3 = \eta_2\underline{\epsilon}_1 + \eta_1\underline{\epsilon}_2 + \underline{\epsilon}_1^\times\underline{\epsilon}_2$$

Notice that no trigonometric function is required, and that we only need to store 4 parameters. Furthermore, the number of operations (multiplication and sum) to compose a rotation is lower than that required by the DCMs.