Quaternions

Quaternions are hypercomplex numbers composed of a real and a vector parts:

$$\underline{Q} = q + \underline{q} = \underbrace{q_x \underline{i} + q_y \underline{j} + q_z \underline{k}}_{\text{Vector part}} + \underbrace{q}_{\text{real or scalar part}}$$

The quaternion algebra states that:

$$\underline{i}^2 = \underline{j}^2 = \underline{k}^2 = -1$$

$$\underline{i}\,\underline{j} = \underline{k} \qquad \underline{j}\,\underline{k} = \underline{i} \qquad \underline{k}\,\underline{i} = \underline{j}$$

The multiplication order matters!

Multiplication table

$\times$	1	$\underline{i}$	$\underline{j}$	$\underline{k}$
1	1	$\underline{i}$	$\underline{j}$	$\underline{k}$
$\underline{i}$	$\underline{i}$	-1	$\underline{k}$	$-\underline{j}$
$\underline{j}$	$\underline{j}$	$-\underline{k}$	-1	$\underline{i}$
$\underline{k}$	$\underline{k}$	$\underline{j}$	$-\underline{i}$	-1

Given that, the multiplication of two quaternions yield:

$$\underline{Q}_1 = \underline{\varepsilon}_1 + \eta_1$$
$$\underline{Q}_2 = \underline{\varepsilon}_2 + \eta_2$$

$$\begin{aligned}
\underline{Q}_1 \underline{Q}_2 &= (\underline{\varepsilon}_1 + \eta_1)(\underline{\varepsilon}_2 + \eta_2) \\
&= \underline{\varepsilon}_1 \underline{\varepsilon}_2 + \eta_1 \eta_2 + \eta_1 \underline{\varepsilon}_2 + \eta_2 \underline{\varepsilon}_1 \\
&= (\varepsilon_{1x} \underline{i} + \varepsilon_{1y} \underline{j} + \varepsilon_{1z} \underline{k})(\varepsilon_{2x} \underline{i} + \varepsilon_{2y} \underline{j} + \varepsilon_{2z} \underline{k}) + \eta_1 \eta_2 + \eta_1 \underline{\varepsilon}_2 + \eta_2 \underline{\varepsilon}_1 \\
&= \underbrace{-\varepsilon_{1x}\varepsilon_{2x} - \varepsilon_{1y}\varepsilon_{2y} - \varepsilon_{1z}\varepsilon_{2z}}_{-\underline{\varepsilon}_1^T \underline{\varepsilon}_2} + \underbrace{(\varepsilon_{1y}\varepsilon_{2z} - \varepsilon_{1z}\varepsilon_{2y})\underline{i} + (\varepsilon_{1x}\varepsilon_{2y} - \varepsilon_{1y}\varepsilon_{2x})\underline{j} + (\varepsilon_{1x}\varepsilon_{2y} - \varepsilon_{1y}\varepsilon_{2x})\underline{k}}_{\underline{\varepsilon}_1^{\times} \underline{\varepsilon}_2} + \eta_1\eta_2 + \eta_1 \underline{\varepsilon}_2 + \eta_2 \underline{\varepsilon}_1
\end{aligned}$$

$$\underline{Q}_1 \underline{Q}_2 = \underbrace{\eta_1 \eta_2 - \underline{\varepsilon}_1^T \underline{\varepsilon}_2}_{\text{Real part}} + \underbrace{\eta_1 \underline{\varepsilon}_2 + \eta_2 \underline{\varepsilon}_1 + \underline{\varepsilon}_1^{\times} \underline{\varepsilon}_2}_{\text{Vector part}}$$

Hence, the result of quaternion multiplication is identical to the result of rotation composition using the Euler symmetric parameters. Thus, we can use quaternions to rotate reference systems.

to change basis

Note: The quaternion multiplicative inverse is the conjugate quaternion $\underline{\bar{Q}} = \eta - \underline{\varepsilon}$:

$$\underline{Q}\,\underline{\bar{Q}} = (\eta + \underline{\varepsilon})(\eta - \underline{\varepsilon}) = \underbrace{\eta^2 + \underline{\varepsilon}^T \underline{\varepsilon}}_{1} + \cancel{\eta \underline{\varepsilon}} - \cancel{\eta \underline{\varepsilon}} + \overbrace{\underline{\varepsilon}^{\times} \underline{\varepsilon}}^{=0} = 1$$

$$\Downarrow$$

$$\underline{Q}^{-1} = \underline{\bar{Q}}, \text{ if } |\underline{Q}| = 1$$

Assuming that the quaternion norm is 1!

Basis transformation using quaternions

Let $\mathfrak{F}_a$ and $\mathfrak{F}_b$ be two reference system related by the Euler symmetric parameters $(\underline{\varepsilon}, \eta)$. Let $\underline{v}$ be a vector in which its representation in $\mathfrak{F}_a$ is known ($\underline{v}_a$). We want to find $\underline{v}_b$ using quaternions.

Consider the following operation: $\longrightarrow$ $\underline{Q}\,\underline{v}_a \triangleq \underline{Q}\,(0+\underline{v}_a)$ Quaternion with real part 0 and vector part $\underline{v}_a$.

$$
\begin{aligned}
\underline{Q}^{-1}\underline{v}_a\underline{Q} &= (\eta-\underline{\varepsilon})(0+\underline{v}_a)(\eta+\underline{\varepsilon}) = (\eta-\underline{\varepsilon})(0\eta - \underline{v}_a^T\underline{\varepsilon} + 0.\underline{\varepsilon} + \eta\underline{v}_a + \underline{v}_a^{\times}\underline{\varepsilon}) \\
&= (\eta-\underline{\varepsilon})(\underbrace{-\underline{v}_a^T\underline{\varepsilon}}_{\text{scalar}} + \underbrace{\eta\underline{v}_a + \underline{v}_a^{\times}\underline{\varepsilon}}_{\text{vector}}) \\
&= -\eta\underline{v}_a^T\underline{\varepsilon} + \underline{\varepsilon}^T(\eta\underline{v}_a + \underline{v}_a^{\times}\underline{\varepsilon}) + \eta(\eta\underline{v}_a + \underline{v}_a^{\times}\underline{\varepsilon}) + \underline{v}_a^T\underline{\varepsilon}\,\underline{\varepsilon} - \underline{\varepsilon}^{\times}(\eta\underline{v}_a + \underline{v}_a^{\times}\underline{\varepsilon}) \\
&= -\eta\underline{v}_a^T\underline{\varepsilon} + \eta\underline{\varepsilon}^T\underline{v}_a + \underline{\varepsilon}^T\underline{v}_a^{\times}\underline{\varepsilon} + \eta^2\underline{v}_a + \eta\underline{v}_a^{\times}\underline{\varepsilon} + \underline{v}_a^T\underline{\varepsilon}\,\underline{\varepsilon} - \eta\underline{\varepsilon}^{\times}\underline{v}_a - \underline{\varepsilon}^{\times}\underline{v}_a^{\times}\underline{\varepsilon} \\
&= -\eta\cancel{\underline{\varepsilon}^T}\underline{v}_a + \eta\cancel{\underline{\varepsilon}^T}\underline{v}_a - \underbrace{\underline{\varepsilon}^T\cancel{\underline{\varepsilon}^{\times}}}_{=0}\underline{v}_a + \eta^2\underline{v}_a - \eta\underline{\varepsilon}^{\times}\underline{v}_a + \underline{\varepsilon}\,\underline{\varepsilon}^T\underline{v}_a - \eta\underline{\varepsilon}^{\times}\underline{v}_a + \underbrace{\underline{\varepsilon}^{\times}\underline{\varepsilon}^{\times}}_{\underline{\varepsilon}\,\underline{\varepsilon}^T - \underline{\varepsilon}^T\underline{\varepsilon}\,\underline{I}_3}\underline{v}_a
\end{aligned}
$$

$$[\underline{v}_a^T\underline{\varepsilon}^{\times T}\underline{\varepsilon}]^T = [-\underline{v}_a^T\underline{\varepsilon}^{\times}\underline{\varepsilon}]^T = 0$$

$$
\begin{aligned}
&= \eta^2\underline{v}_a - 2\eta\underline{\varepsilon}^{\times}\underline{v}_a + \underline{\varepsilon}\,\underline{\varepsilon}^T\underline{v}_a + \underline{\varepsilon}\,\underline{\varepsilon}^T\underline{v}_a - \underline{\varepsilon}^T\underline{\varepsilon}\,\underline{v}_a \\
&= (\eta^2 - \underline{\varepsilon}^T\underline{\varepsilon})\underline{v}_a + 2\underline{\varepsilon}\,\underline{\varepsilon}^T\underline{v}_a - 2\eta\underline{\varepsilon}^{\times}\underline{v}_a \\
&= [(\eta^2 - \underline{\varepsilon}^T\underline{\varepsilon}) + 2\underline{\varepsilon}\,\underline{\varepsilon}^T - 2\eta\underline{\varepsilon}^{\times}]\underline{v}_a
\end{aligned}
$$

Thus:

$$\underline{Q}^{-1}\underline{v}_a\underline{Q} = \underbrace{[(\eta^2 - \underline{\varepsilon}^T\underline{\varepsilon}) + 2\underline{\varepsilon}\,\underline{\varepsilon}^T - 2\eta\underline{\varepsilon}^{\times}]}_{\underline{C}_{ba}}\underline{v}_a$$

> **Note**: The real part of the quaternion $\bar{\underline{Q}}\,\underline{v}_a\underline{Q}$ will always be 0.

$$\underline{Q}^{-1}\underline{v}_a\underline{Q} = \bar{\underline{Q}}\,\underline{v}_a\underline{Q} = \underline{C}_{ba}\underline{v}_a = \underline{v}_b$$

$$\underline{v}_b = \bar{\underline{Q}}\,\underline{v}_a\underline{Q}$$

Rotation composition using quaternions

Let $\underline{Q}_{ba}$ be the quaternion that transforms $\mathfrak{F}_a$ into $\mathfrak{F}_b$, and $\underline{Q}_{cb}$ be the quaternion that transforms $\mathfrak{F}_b$ into $\mathfrak{F}_c$. Thus, the quaternion that transforms $\mathfrak{F}_a$ into $\mathfrak{F}_c$ can be computed using:

$$
\left.
\begin{aligned}
\underline{v}_b &= \bar{\underline{Q}}_{ba}\underline{v}_a\underline{Q}_{ba} \\
\underline{v}_c &= \bar{\underline{Q}}_{cb}\underline{v}_b\underline{Q}_{cb}
\end{aligned}
\right. , \quad \forall\, \underline{v} \in \mathbb{R}^3
$$

$$\underline{v}_c = \bar{\underline{Q}}_{cb}\bar{\underline{Q}}_{ba}\underline{v}_a\underline{Q}_{ba}\underline{Q}_{cb}$$

However, $\underline{v}_c = \underline{Q}_{ca}\underline{v}_a\bar{\underline{Q}}_{ca}$. Since this result is valid $\forall\, \underline{v} \in \mathbb{R}^3$, then:

$\underline{Q}_{ca} = \underline{Q}_{ba}\underline{Q}_{cb}$ $\longrightarrow$ Notice that the multiplication order is inverse of that of the DCMs.