Infinitesimal rotations

Let's consider the rotation between $\underrightarrow{\mathcal{F}}_a$ and $\underrightarrow{\mathcal{F}}_b$ in which the Euler axis is $\underline{v}$ and the Euler angle θ is small.

$$\underline{C}_{ba} = \cos\theta\, \underline{I}_3 + (1-\cos\theta)\underline{v}\,\underline{v}^T - \sin\theta\, \underline{v}^\times$$

Since θ is small, then the following approximation holds:

$$\sin\theta \approx \theta \qquad \cos\theta \approx 1$$

Thus:

$$\underline{C}_{ba} \approx \underline{I}_3 - \theta\, \underline{v}^\times$$

Now, let's consider that the rotation between $\underrightarrow{\mathcal{F}}_b$ and $\underrightarrow{\mathcal{F}}_c$ is also small. Hence:

$$\underline{C}_{ca} = \underline{C}_{cb}\underline{C}_{ba} \approx [\underline{I}_3 - \theta_2 \underline{v}_2^\times][\underline{I}_3 - \theta_1 \underline{v}_1^\times]$$

$$\underline{C}_{ca} \approx \underline{I}_3 - \theta_2 \underline{v}_2^\times - \theta_1 \underline{v}_1^\times + \underbrace{\theta_1\theta_2}_{\approx 0} \underline{v}_2^\times \underline{v}_1$$

$$\underline{C}_{ca} \approx \underline{I}_3 - \theta_2 \underline{v}_2^\times - \theta_1 \underline{v}_1^\times$$

↳ Notice that in this case the order of the rotations does not matter.

Finally, using Euler theorem, let's decompose the rotation between $\underrightarrow{\mathcal{F}}_a$ and $\underrightarrow{\mathcal{F}}_b$ using three rotations of θ_1, θ_2, and θ_3 about the coordinate axes:

$$\underline{C}_{ba} = \underline{R}_3(\theta_3)\underline{R}_2(\theta_2)\underline{R}_1(\theta_1) \approx \underline{I}_3 - \theta_3 \underline{e}_3^\times - \theta_2 \underline{e}_2^\times - \theta_1 \underline{e}_1^\times, \quad \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus:

$$\underline{C}_{ba} = \underline{I}_3 - \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ +\theta_3 & 0 & -\theta_1 \\ -\theta_2 & +\theta_1 & 0 \end{bmatrix} = \underline{I}_3 - \underline{\theta}^\times, \quad \underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

Finally, for small rotations:

$$\underline{C}_{ba} = \underline{I}_3 - \underline{\theta}^\times, \quad \underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$