Kinematics

We want to describe how the attitude of an object changes with time given its velocity and acceleration.

Angular velocity

Definition: instantaneous vector about which the object rotates.

Considering infinitesimal angles:

$$\underline{\omega} = \underline{\dot{\theta}} \quad , \quad \underline{\theta} = [\,\theta_1 \quad \theta_2 \quad \theta_3\,]^T$$

Because of the Euler angle and axis representation, we can conclude that the angular velocity vector is always aligned with the instateneous Euler axis and its magnitude is given by the rate of change of the Euler angle.

We can interpret the angular velocity as the measurement of the motion of an object (using the reference sytem that describes its attitude) with respect to another reference system. Thus, we call $\vec{\omega}_{ba}$ as the angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_a$. Notice that this vector is the same for observers in $\mathcal{F}_a$ and $\mathcal{F}_b$.

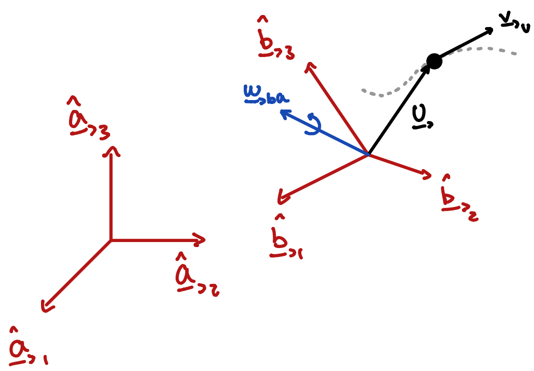
This does not mean that $\vec{\omega}_{ba}$ has the same representation in $\mathcal{F}_a$ and $\mathcal{F}_b$! Only that observers in both reference systems measure the same vector.

Hence, $\vec{\omega}_{ab}$ is the angular velocity of $\mathcal{F}_a$ with respect to $\mathcal{F}_b$ and we can write:

$$\vec{\omega}_{ba} + \vec{\omega}_{ab} = \vec{0}$$

Movement composition between systems

We want to obtain the position vector $\vec{v}$ of a particle with respect to the reference system $\mathcal{F}_a$ when the representation of $\vec{v}$ is only known in the reference system $\mathcal{F}_b$ that has angular motion with respect to $\mathcal{F}_a$.



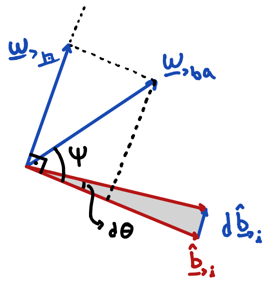
First, suppose that $\vec{v}$ does not move with respect to $\mathcal{F}_b$. Hence:

$$\dot{\underrightarrow{u}} = (\underrightarrow{\mathcal{F}}_b^T u_b)^{\cdot} = \dot{\underrightarrow{\mathcal{F}}}_b^T u_b + \underrightarrow{\mathcal{F}}_b^T \cancel{\dot{u}_b} = \dot{\underrightarrow{\mathcal{F}}}_b^T u_b$$

$$\dot{\underrightarrow{u}} = [\dot{\hat{\underrightarrow{b}}}_1 \ \ \dot{\hat{\underrightarrow{b}}}_2 \ \ \dot{\hat{\underrightarrow{b}}}_3]\, u_b$$

u_b is constant because we are supposing that $\underrightarrow{u}$ does not move with respect to $\underrightarrow{\mathcal{F}}_b$.

Notice that:



1) $|\underrightarrow{\omega}_{\perp}| = |\underrightarrow{\omega}_{ba}| \sin\Psi$

2) $d\theta = |\underrightarrow{\omega}_{\perp}| \cdot dt = |\underrightarrow{\omega}_{ba}| \sin\Psi \, dt \Rightarrow \dfrac{d\theta}{dt} = |\underrightarrow{\omega}_{ba}| \sin\Psi$

Since $\hat{\underrightarrow{b}}_i$ is unitary, then:

$$|d\underrightarrow{b}_i| = d\theta \Rightarrow |\dot{\hat{\underrightarrow{b}}}_i| = \left|\frac{d\hat{\underrightarrow{b}}_i}{dt}\right| = \frac{d\theta}{dt} = |\underrightarrow{\omega}_{ba}| \sin\Psi = |\underrightarrow{\omega}_{\perp}| = |\underrightarrow{\omega}_{ba} \times \hat{\underrightarrow{b}}_i|$$

Moreover, notice that the direction of $\dot{\hat{\underrightarrow{b}}}_i$ is the same of the vector $\underrightarrow{\omega}_{ba} \times \hat{\underrightarrow{b}}_i$. Hence:

$$\dot{\hat{\underrightarrow{b}}}_i = \underrightarrow{\omega}_{ba} \times \hat{\underrightarrow{b}}_i \ , \quad i \in [1, 2, 3]$$

Using this result, one can see that:

$$\dot{\underrightarrow{u}} = [\dot{\hat{\underrightarrow{b}}}_1 \ \ \dot{\hat{\underrightarrow{b}}}_2 \ \ \dot{\hat{\underrightarrow{b}}}_3]\, u_b = [\underrightarrow{\omega}_{ba} \times \hat{\underrightarrow{b}}_1 \ \ \underrightarrow{\omega}_{ba} \times \hat{\underrightarrow{b}}_2 \ \ \underrightarrow{\omega}_{ba} \times \hat{\underrightarrow{b}}_3]\, u_b$$

$$\dot{\underrightarrow{u}} = \underrightarrow{\omega}_{ba} \times \underbrace{[\underrightarrow{b}_1 \ \ \underrightarrow{b}_2 \ \ \underrightarrow{b}_3]}_{\underrightarrow{\mathcal{F}}_b^T} u_b$$

$$\dot{\underrightarrow{u}} = \underrightarrow{\omega}_{ba} \times \overbrace{\underrightarrow{\mathcal{F}}_b^T u_b}^{\underrightarrow{u}}$$

$$\dot{\underrightarrow{u}} = \underrightarrow{\omega}_{ba} \times \underrightarrow{u}$$

Finally, if $\underrightarrow{u}$ is fixed in $\underrightarrow{\mathcal{F}}_b$, then:

$$\underrightarrow{v}_v = \frac{d}{dt}\underrightarrow{u} = \dot{\underrightarrow{u}} = \underrightarrow{\omega}_{ba} \times \underrightarrow{u}$$

If $\underrightarrow{u}$ moves with respect to $\underrightarrow{\mathcal{F}}_b$, then:

$$\dot{\underrightarrow{u}} = (\underrightarrow{\mathcal{F}}_b^T u_b)^{\cdot} = \dot{\underrightarrow{\mathcal{F}}}_b^T u_b + \underbrace{\boxed{\underrightarrow{\mathcal{F}}_b^T \dot{u}_b}}_{\left.\frac{d}{dt}\underrightarrow{u}\right|_b}$$

$\left.\frac{d}{dt}\underrightarrow{u}\right|_b$ → velocity of $\underrightarrow{u}$ measured by an observer in $\underrightarrow{\mathcal{F}}_b$

$$\dot{\underrightarrow{u}} = \underrightarrow{\omega}_{ba} \times \underrightarrow{u} + \left.\frac{d}{dt}\underrightarrow{u}\right|_b$$

Thus, it is now clear that the velocity of $\underrightarrow{u}$ depends on the reference system in which the velocity is being measured. We will use the following notation to indicate this fact:

$$\left.\frac{d}{dt}\underrightarrow{u}\right|_a = \left.\frac{d}{dt}\underrightarrow{u}\right|_b + \underrightarrow{\omega}_{ba} \times \underrightarrow{u}$$

To simplify the notation, the following notation will be used:

$$\dot{\underrightarrow{u}}^a \triangleq \left.\frac{d}{dt}\underrightarrow{u}\right|_a \qquad \dot{\underrightarrow{u}}^b \triangleq \left.\frac{d}{dt}\underrightarrow{u}\right|_b$$

Hence:

$$\dot{\underrightarrow{u}}^a = \dot{\underrightarrow{u}}^b + \underrightarrow{\omega}_{ba} \times \underrightarrow{u}$$

This is the relationship of the particle velocity measured in $\underrightarrow{\mathcal{F}}_b$ with the particle velocity in $\underrightarrow{\mathcal{F}}_a$, in which the angular velocity of $\underrightarrow{\mathcal{F}}_b$ with respect to $\underrightarrow{\mathcal{F}}_a$ is $\underrightarrow{\omega}_{ba}$.

Now, let $\underrightarrow{\mathcal{F}}_c$ be another reference system with angular velocity $\underrightarrow{\omega}_{cb}$ with respect to $\underrightarrow{\mathcal{F}}_b$. Thus:

$$\dot{\underrightarrow{u}}^b = \dot{\underrightarrow{u}}^c + \underrightarrow{\omega}_{cb} \times \underrightarrow{u}$$

Hence:

$$\begin{aligned}\dot{\underrightarrow{u}}^a &= \dot{\underrightarrow{u}}^b + \underrightarrow{\omega}_{ba} \times \underrightarrow{u}\\ &= \dot{\underrightarrow{u}}^c + \underrightarrow{\omega}_{cb} \times \underrightarrow{u} + \underrightarrow{\omega}_{ba} \times \underrightarrow{u}\end{aligned}$$

$$\dot{\underrightarrow{u}}^a = \dot{\underrightarrow{u}}^c + (\underrightarrow{\omega}_{cb} + \underrightarrow{\omega}_{ba}) \times \underrightarrow{u}$$

Since:

$$\dot{\underrightarrow{u}}^a = \dot{\underrightarrow{u}}^c + \underrightarrow{\omega}_{ca} \times \underrightarrow{u}$$

Then:

$$\underrightarrow{\omega}_{ca} = \underrightarrow{\omega}_{cb} + \underrightarrow{\omega}_{ba}$$

The acceleration of the vector $\underrightarrow{u}$ can be computed by:

$$\ddot{\underrightarrow{u}}^a \triangleq \left.\frac{d}{dt}\left(\dot{\underrightarrow{u}}^a\right)\right|_a = \left.\frac{d}{dt}\left(\dot{\underrightarrow{u}}^b + \underrightarrow{\omega}_{ba} \times \underrightarrow{u}\right)\right|_a = \left.\frac{d}{dt}\dot{\underrightarrow{u}}^b\right|_a + \left.\frac{d}{dt}\underrightarrow{\omega}_{ba}\right|_a \times \underrightarrow{u} + \underrightarrow{\omega}_{ba} \times \left.\frac{d}{dt}\underrightarrow{u}\right|_a$$

using the previous result for each vector

$$\ddot{\underrightarrow{u}}^a = \underbrace{\left.\frac{d}{dt}\dot{\underrightarrow{u}}^b\right|_b}_{\ddot{\underrightarrow{u}}^b} + \underrightarrow{\omega}_{ba} \times \dot{\underrightarrow{u}}^b + \left[\underbrace{\left.\frac{d}{dt}\underrightarrow{\omega}_{ba}\right|_b}_{\dot{\underrightarrow{\omega}}^b_{ba}} + \cancel{\underrightarrow{\omega}_{ba} \times \underrightarrow{\omega}_{ba}}\right] \times \underrightarrow{u} + \underrightarrow{\omega}_{ba} \times \left[\underbrace{\left.\frac{d}{dt}\underrightarrow{u}\right|_b}_{\dot{\underrightarrow{u}}^b} + \underrightarrow{\omega}_{ba} \times \underrightarrow{u}\right]$$

$$\ddot{\underrightarrow{u}}^a = \ddot{\underrightarrow{u}}^b + \underrightarrow{\omega}_{ba} \times \dot{\underrightarrow{u}}^b + \dot{\underrightarrow{\omega}}^b_{ba} \times \underrightarrow{u} + \underrightarrow{\omega}_{ba} \times \dot{\underrightarrow{u}}^b + \underrightarrow{\omega}_{ba} \times (\underrightarrow{\omega}_{ba} \times \underrightarrow{u})$$

Thus:

$$\ddot{\underrightarrow{u}}^a = \underbrace{\ddot{\underrightarrow{u}}^b}_{\text{Acceleration of the particle measured by an observer in } \underrightarrow{\mathcal{F}}_b} + \underbrace{2\underrightarrow{\omega}_{ba} \times \dot{\underrightarrow{u}}^b}_{\text{Coriolis acceleration}} + \underbrace{\dot{\underrightarrow{\omega}}^b_{ba} \times \underrightarrow{u}}_{\text{Angular acceleration of } \underrightarrow{\mathcal{F}}_b \text{ with respect to } \underrightarrow{\mathcal{F}}_a} + \underbrace{\underrightarrow{\omega}_{ba} \times (\underrightarrow{\omega}_{ba} \times \underrightarrow{u})}_{\text{Centripetal acceleration}}$$

Note: Time-derivative of vectrizes

Using the same result, one can see that:

$$\dot{\underrightarrow{\mathcal{F}}}^a_b = \begin{bmatrix} \dot{\hat{\underrightarrow{b}}}^a_1 & \dot{\hat{\underrightarrow{b}}}^a_2 & \dot{\hat{\underrightarrow{b}}}^a_3 \end{bmatrix}^T = \underrightarrow{\omega}_{ba} \times \underrightarrow{\mathcal{F}}_b$$