Kinematics using DCMs

We want to verify how the attitude representation using DCMs varies in time when the reference system $\underrightarrow{\mathcal{F}}_b$ has angular velocity $\underrightarrow{\omega}_{ba}$ with respect to $\underrightarrow{\mathcal{F}}_a$.

Notice that:

$$C_{ba} = \underrightarrow{\mathcal{F}}_b \cdot \underrightarrow{\mathcal{F}}_a^T$$

Hence:

$$
\begin{aligned}
\frac{d}{dt} C_{ba}\Big|_a &= \frac{d}{dt}\underrightarrow{\mathcal{F}}_b\Big|_a \cdot \underrightarrow{\mathcal{F}}_a^T + \underrightarrow{\mathcal{F}}_b \cdot \cancelto{\underrightarrow{0}}{\frac{d}{dt}\underrightarrow{\mathcal{F}}_a^T\Big|_a} = \overset{\underrightarrow{\mathcal{F}}_b^T \omega_{ba,b}}{\overset{\shortparallel}{\underrightarrow{\omega}_{ba}}} \times \underrightarrow{\mathcal{F}}_b \cdot \underrightarrow{\mathcal{F}}_a^T \\
&= \left(\underrightarrow{\mathcal{F}}_b^T \omega_{ba,b}\right) \times \underrightarrow{\mathcal{F}}_b \cdot \underrightarrow{\mathcal{F}}_a^T \\
&= -\left[\underrightarrow{\mathcal{F}}_b \times \left(\underrightarrow{\mathcal{F}}_b^T \omega_{ba,b}\right)\right] \cdot \underrightarrow{\mathcal{F}}_a^T \\
&= -\left[\underbrace{\underrightarrow{\mathcal{F}}_b \times \underrightarrow{\mathcal{F}}_b^T}_{-\underrightarrow{\mathcal{F}}_b^\times} \omega_{ba,b}\right] \cdot \underrightarrow{\mathcal{F}}_a^T \\
&= +\underrightarrow{\mathcal{F}}_b^\times \omega_{ba,b} \cdot \underrightarrow{\mathcal{F}}_a^T \\
&= -\omega_{ba,b}^\times \underbrace{\underrightarrow{\mathcal{F}}_b \cdot \underrightarrow{\mathcal{F}}_a^T}_{C_{ba}} \\
&= -\omega_{ba,b}^\times C_{ba}
\end{aligned}
$$

Futhermore:

$$
\begin{aligned}
\frac{d}{dt} C_{ba}\Big|_b &= \cancelto{\underrightarrow{0}}{\frac{d}{dt}\underrightarrow{\mathcal{F}}_b\Big|_b} \cdot \underrightarrow{\mathcal{F}}_a^T + \underrightarrow{\mathcal{F}}_b \cdot \frac{d}{dt}\underrightarrow{\mathcal{F}}_a^T\Big|_b = \underrightarrow{\mathcal{F}}_b \cdot \left(\underrightarrow{\omega}_{ab} \times \underrightarrow{\mathcal{F}}_a\right)^T \\
&= -\underrightarrow{\mathcal{F}}_b \cdot \Big(\underrightarrow{\mathcal{F}}_a \times \underset{\underrightarrow{\mathcal{F}}_a \omega_{ab,a}}{\underset{\shortparallel}{\underrightarrow{\omega}_{ab}}}\Big)^T \\
&= -\underrightarrow{\mathcal{F}}_b \cdot \Big(\underbrace{\underrightarrow{\mathcal{F}}_a \times \underrightarrow{\mathcal{F}}_a}_{-\underrightarrow{\mathcal{F}}_a^\times} \omega_{ab,a}\Big)^T \\
&= +\underrightarrow{\mathcal{F}}_b \cdot \left(\underrightarrow{\mathcal{F}}_a^\times \omega_{ab,a}\right)^T \\
&= -\underrightarrow{\mathcal{F}}_b \cdot \left(\omega_{ab,a}^\times \underrightarrow{\mathcal{F}}_a\right)^T \\
&= -\underbrace{\underrightarrow{\mathcal{F}}_b \cdot \underrightarrow{\mathcal{F}}_a^T}_{C_{ba}} \underset{-\omega_{ab,a}^\times}{\underset{\shortparallel}{\omega_{ab,a}^{\times T}}} \\
&= +C_{ba}\, \omega_{ab,a}^\times \\
&= +\underbrace{C_{ab}^T\, \omega_{ab,a}^\times \overbrace{C_{ab}\, C_{ab}^T}^{I_3}}_{\left(C_{ab}^T \omega_{ab,a}\right)^\times = \left(C_{ba}\omega_{ab,a}\right)^\times = \omega_{ab,b}^\times} \\
&= \underset{-\omega_{ba,b}^\times}{\underset{\shortparallel}{\omega_{ab,b}^\times}}\, C_{ba} = -\omega_{ba,b}^\times C_{ba} \qquad \left(\underrightarrow{\omega}_{ba} + \underrightarrow{\omega}_{ab} = \underrightarrow{0}\right)
\end{aligned}
$$

Thus, the time-derivative of C_{ba} is the same in $\underrightarrow{\mathcal{F}}_a$ and $\underrightarrow{\mathcal{F}}_b$. Hence:

$$\dot{C}_{ba} = -\omega_{ba,b}^\times C_{ba} = C_{ba}\,\omega_{ab,a}^\times = -C_{ba}\,\omega_{ba,a}^\times$$

Example: Attitude propagation of a satellite without external torques

Suppose that:

1) A satellite is in space free from external torques.
2) We know the initial angular velocity represented in the inertial reference system.
   ↳ The angular velocity is constant because there are no external torques.
3) We know the initial attitude.

We want to compute the satellite attitude at every instant.

Initial attitude

ϕ = roll $\qquad \theta$ = pitch $\qquad \psi$ = yaw

$$\underline{C}_{bi}(0) = \underline{C}_1(\phi)\,\underline{C}_2(\theta)\,\underline{C}_3(\psi)$$

Attitude propagation

$$\underline{\omega}_{bi,b}(t) = \underline{C}_{bi}(t)\,\underline{\omega}_{bi,i}$$

↳ This is what the gyros measure.

$$\dot{\underline{C}}_{bi} = -\underline{\omega}_{bi,b}^{\times}(t)\,\underline{C}_{bi}(t)$$

Finally, we only need an algorithm to solve numerically that EDO obtaining $\underline{C}_{bi}(t)$. The most simple (and less accurate) is the Euler backward method:

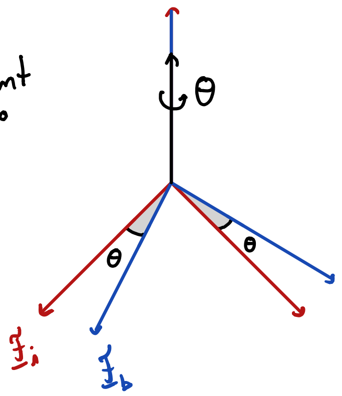
$$\underline{C}_{bi}(t_k) = \underline{C}_{bi}(t_{k-1}) - \underline{\omega}_{bi,b}(t_{k-1})\,\underline{C}_{bi}(t_{k-1})\cdot(t_k - t_{k-1}),$$ where $t_k = t_{k-1} + \Delta$ and Δ is the sampling interval.

$$\underline{C}_{bi}(k) = \underline{C}_{bi}(k-1) - \underline{\omega}_{bi,b}(k-1)\,\underline{C}_{bi}(k-1)\cdot\Delta$$
$$\underline{C}_{bi}(0) = \underline{C}_1(\phi)\,\underline{C}_2(\theta)\,\underline{C}_3(\psi)$$

Example: A simplified case

Consider the following example:

$\dot{\theta}$ constant
$\theta(0) = 0°$



Thus:

$$\underline{\omega}_{bi,b} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \qquad \underline{C}_{bi}(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dot{\underline{C}}_{bi,b} = -\underline{\omega}_{bi,b}^{\times}\,\underline{C}_{bi,b} = \begin{bmatrix} 0 & +\dot{\theta} & 0 \\ -\dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$\dot{\underline{C}}_{bi,b} = \begin{bmatrix} +\dot{\theta} C_{21} & +\dot{\theta} C_{22} & +\dot{\theta} C_{23} \\ -\dot{\theta} C_{11} & -\dot{\theta} C_{12} & -\dot{\theta} C_{13} \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, we have the following system of EDO's:

$$\begin{bmatrix} \dot{C}_{11} \\ \dot{C}_{12} \\ \dot{C}_{13} \\ \dot{C}_{21} \\ \dot{C}_{22} \\ \dot{C}_{23} \\ \dot{C}_{31} \\ \dot{C}_{32} \\ \dot{C}_{33} \end{bmatrix} = \begin{bmatrix} +\dot{\theta} C_{21} \\ +\dot{\theta} C_{22} \\ +\dot{\theta} C_{23} \\ -\dot{\theta} C_{11} \\ -\dot{\theta} C_{12} \\ -\dot{\theta} C_{13} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

It is clear that:

$C_{31}(t) = K_1$

$C_{32}(t) = K_2$

$C_{33}(t) = K_3$

Hence, using the initial value:

$C_{31}(t) = 0$

$C_{32}(t) = 0$

$C_{33}(t) = 1$

This system can be split into:

$$\begin{bmatrix} \dot{C}_{11} \\ \dot{C}_{21} \end{bmatrix} = \begin{bmatrix} 0 & +\dot{\theta} \\ -\dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} \qquad \begin{bmatrix} \dot{C}_{12} \\ \dot{C}_{22} \end{bmatrix} = \begin{bmatrix} 0 & +\dot{\theta} \\ -\dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} C_{12} \\ C_{22} \end{bmatrix} \qquad \begin{bmatrix} \dot{C}_{13} \\ \dot{C}_{23} \end{bmatrix} = \begin{bmatrix} 0 & +\dot{\theta} \\ -\dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} C_{13} \\ C_{23} \end{bmatrix}$$

Using the theory of EDO:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & +\dot{\theta} \\ -\dot{\theta} & 0 \end{bmatrix} \underline{x} \quad \Rightarrow \quad \underline{x}(t) = e^{\underline{A}t} \underline{x}(0)$$

$$e^{\underline{A}t} = \underline{I}_3 + \underline{A}t + \frac{1}{2!} \underline{A}^2 t^2 + \frac{1}{3!} \underline{A}^3 t^3 + \dots$$

Notice that:

$$\underline{A}^2 = \begin{bmatrix} -\dot{\theta}^2 & 0 \\ 0 & -\dot{\theta}^2 \end{bmatrix} \quad \underline{A}^3 = \begin{bmatrix} 0 & -\dot{\theta}^3 \\ +\dot{\theta}^3 & 0 \end{bmatrix} \quad \underline{A}^4 = \begin{bmatrix} +\dot{\theta}^4 & 0 \\ 0 & +\dot{\theta}^4 \end{bmatrix} \quad \underline{A}^5 = \begin{bmatrix} 0 & +\dot{\theta}^5 \\ -\dot{\theta}^5 & 0 \end{bmatrix}$$

$$\Rightarrow A^K = \begin{cases} \begin{bmatrix} (-1)^{\frac{K}{2}} \dot{\theta}^K & 0 \\ 0 & (-1)^{\frac{K}{2}} \dot{\theta}^K \end{bmatrix}, & \text{if } K = 2n,\ n \in \mathbb{N} \\ \begin{bmatrix} 0 & (-1)^{\frac{K-1}{2}} \dot{\theta}^K \\ (-1)^{\frac{K+1}{2}} \dot{\theta}^K & 0 \end{bmatrix}, & \text{if } K = 2n+1,\ n \in \mathbb{N} \end{cases}$$

Thus:

$$e^{At} = \left[\begin{array}{c:c} \overbrace{1 - \frac{1}{2!}\dot{\theta}^2 t^2 + \frac{1}{4!}\dot{\theta}^4 t^4 - \dots}^{\cos(\dot{\theta}t)} & \overbrace{+\dot{\theta} - \frac{1}{3!}\dot{\theta}^3 t^3 - \frac{1}{5!}\dot{\theta}^5 t^5 + \dots}^{+\sin(\dot{\theta}t)} \\ \hdashline \underbrace{-\dot{\theta} + \frac{1}{3!}\dot{\theta}^3 t^3 - \frac{1}{5!}\dot{\theta}^5 t^5 + \dots}_{-\sin(\dot{\theta}t)} & \underbrace{1 - \frac{1}{2!}\dot{\theta}^2 t^2 + \frac{1}{4!}\dot{\theta}^4 t^4 - \dots}_{\cos(\dot{\theta}t)} \end{array}\right]$$

Hence:

$$\underline{x}(t) = \begin{bmatrix} \cos(\dot{\theta}t) & \sin(\dot{\theta}t) \\ -\sin(\dot{\theta}t) & \cos(\dot{\theta}t) \end{bmatrix} \underline{x}(0)$$

Which leads to:

$$\begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} = \begin{bmatrix} \cos(\dot{\theta}t) & \sin(\dot{\theta}t) \\ -\sin(\dot{\theta}t) & \cos(\dot{\theta}t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\dot{\theta}t) \\ -\sin(\dot{\theta}t) \end{bmatrix}$$

$$\begin{bmatrix} C_{12} \\ C_{22} \end{bmatrix} = \begin{bmatrix} \cos(\dot{\theta}t) & \sin(\dot{\theta}t) \\ -\sin(\dot{\theta}t) & \cos(\dot{\theta}t) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin(\dot{\theta}t) \\ \cos(\dot{\theta}t) \end{bmatrix}$$

$$\begin{bmatrix} C_{13} \\ C_{33} \end{bmatrix} = \begin{bmatrix} \cos(\dot{\theta}t) & \sin(\dot{\theta}t) \\ -\sin(\dot{\theta}t) & \cos(\dot{\theta}t) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Finally:

$$\underline{C}_{bi}(t) = \begin{bmatrix} \cos(\dot{\theta}t) & \sin(\dot{\theta}t) & 0 \\ -\sin(\dot{\theta}t) & \cos(\dot{\theta}t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$