# Kinematics using quaternions

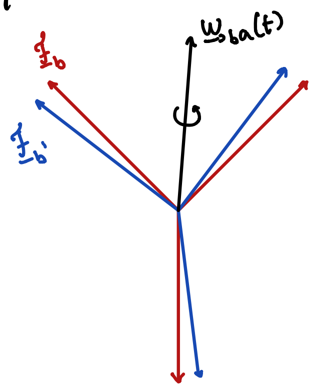
We want to verify how the attitude representation using quaternions varies in time when the reference system $\mathcal{F}_b$ has angular velocity $\underline{\omega}_{ba}$ with respect to $\mathcal{F}_a$. Hence, at instant $t$, the attitude is represented by $\underline{Q}_{ba}(t)$ and the angular velocity is $\underline{\omega}_{ba}(t)$.

For very small $\Delta t$, the reference system $\mathcal{F}_b$ is rotated to $\mathcal{F}_{b'}$. The instateneous rotation axis is given by $\underline{\omega}_{ba}(t)$. Hence:

$$\underline{Q}_{ba}(t+\Delta t) = \underline{Q}_{ba}(t)\, \underline{Q}_{b'b} = \underline{Q}_{ba}(t)\, \Delta \underline{Q}$$

$\longrightarrow$ $\mathcal{F}_{b'}$ is the reference system $\mathcal{F}_b$ in the instant $t+\Delta t$

The quaternion $\Delta \underline{Q}$ can be computed as follows:




$$\Delta \underline{Q} = \Delta \eta + \Delta \underline{\varepsilon}$$

$$\Delta \eta = \cos \frac{\Delta\theta}{2} \qquad \Delta \underline{\varepsilon} = \overbrace{\boxed{\frac{\underline{\omega}_{ba,b}}{|\underline{\omega}_{ba,b}|}}}^{\hat{\underline{\omega}}_{ba,b}} \sin \frac{\Delta\theta}{2}$$

$\hookrightarrow$ must be represented in $\mathcal{F}_b$ instead of $\mathcal{F}_a$ because $\Delta \underline{Q}$ is a small rotation from $\mathcal{F}_b$.

Where $\Delta\theta$ is the magnitude of the rotation, which is given by:

$$\Delta\theta = |\underline{\omega}_{ba,b}|\, \Delta t = \omega \Delta t$$

Hence, notice that:

$$\underline{Q}_{ba}(t+\Delta t) - \underline{Q}_{ba}(t) = \underline{Q}_{ba}(t)\,\Delta \underline{Q} - \underline{Q}_{ba}(t) = \underline{Q}_{ba}(t)\,(\Delta \underline{Q} - 1)$$

$$\underline{Q}_{ba}(t+\Delta t) - \underline{Q}_{ba}(t) = \underline{Q}_{ba}(t) \left[ -1 + \cos \frac{\omega \Delta t}{2} + \hat{\underline{\omega}}_{ba,b} \sin \frac{\omega \Delta t}{2} \right]$$

From the definition of derivative:

$$\dot{\underline{Q}}_{ba}(t) \triangleq \lim_{\Delta t \to 0} \frac{\underline{Q}_{ba}(t+\Delta t) - \underline{Q}_{ba}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\underline{Q}_{ba}(t)}{\Delta t} \left[ -1 + \cos \frac{\omega \Delta t}{2} + \hat{\underline{\omega}}_{ba,b} \sin \frac{\omega \Delta t}{2} \right]$$

$$\dot{\underline{Q}}_{ba}(t) = \underline{Q}_{ba}(t) \cdot \left[ \lim_{\Delta t \to 0} \frac{1 - \cos \frac{\omega \Delta t}{2}}{\Delta t} + \hat{\underline{\omega}}_{ba,b} \lim_{\Delta t \to 0} \frac{\sin \frac{\omega \Delta t}{2}}{\Delta t} \right]$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\lim_{\Delta t \to 0} \frac{\omega}{2} \sin \frac{\omega \Delta t}{2} = 0 \qquad\qquad \lim_{\Delta t \to 0} \frac{\omega}{2} \cos \frac{\omega \Delta t}{2} = \frac{\omega}{2}$$

Thus:

$$\dot{\underline{Q}}_{ba}(t) = \underline{Q}_{ba}(t)\,\hat{\underline{\omega}}_{ba,b}\,\frac{\omega}{2} = \underline{Q}_{ba}\,\frac{\underline{\omega}_{ba,b}}{\cancel{|\underline{\omega}_{ba,b}|}}\cdot\cancel{|\underline{\omega}_{ba,b}|}$$

Finally: Quaternion multiplication!

$$\dot{\underline{Q}}_{ba}(t) = \frac{1}{2}\,\underline{Q}_{ba}(t)\,\underline{\omega}_{ba,b}$$

If $\underline{Q}_{ba} = \eta + \underline{\varepsilon}$, then:

$$\dot{\underline{Q}}_{ba}(t) = \frac{1}{2}\left(-\underline{\varepsilon}^{T}\underline{\omega}_{ba,b} + \eta\,\underline{\omega}_{ba,b} + \underline{\varepsilon}^{\times}\underline{\omega}_{ba,b}\right)$$

$$= \frac{1}{2}\Big(\underbrace{-\underline{\omega}_{ba,b}^{T}\underline{\varepsilon}}_{\text{scalar}} + \underbrace{\eta\,\underline{\omega}_{ba,b} - \underline{\omega}_{ba,b}^{\times}\underline{\varepsilon}}_{\text{vectorial}}\Big)$$

Thus, if we define the quaternion as a column matrix:

$$\underline{Q}_{ba} = \begin{pmatrix}\underline{\varepsilon}\\ \eta\end{pmatrix}$$

then:

$$\begin{pmatrix}\dot{\underline{\varepsilon}}\\ \dot{\eta}\end{pmatrix} = \frac{1}{2}\left[\begin{array}{c:c} -\underline{\omega}_{ba,b}^{\times} & \underline{\omega}_{ba,b} \\ \hdashline -\underline{\omega}_{ba,b}^{T} & 0\end{array}\right]\begin{pmatrix}\underline{\varepsilon}\\ \eta\end{pmatrix}$$

Finally, the dynamic equation of the quaternion is:

$$\dot{\underline{Q}}_{ba} = \frac{1}{2}\,\underline{\Omega}_{ba,b}\,\underline{Q}_{ba}$$

Notice that this is a linear equation!

where:

- $\underline{Q}_{ba} = \begin{pmatrix}\underline{\varepsilon}\\ \eta\end{pmatrix}$
- $\underline{\Omega}_{ba,b} \triangleq \left[\begin{array}{c:c} -\underline{\omega}_{ba,b}^{\times} & \underline{\omega}_{ba,b} \\ \hdashline -\underline{\omega}_{ba,b}^{T} & 0\end{array}\right] = \left[\begin{array}{ccc:c} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ \hdashline -\omega_1 & -\omega_2 & -\omega_3 & 0\end{array}\right]$, if $\underline{\omega}_{ba,b} = (\omega_1 \quad \omega_2 \quad \omega_3)^T$

If the angular velocity is represented in $\mathcal{F}_a$, then:

$$\dot{\underline{Q}}_{ba}(t) = \frac{1}{2}\,\underline{Q}_{ba}(t)\,\underline{\omega}_{ba,b}(t) = \frac{1}{2}\,\overset{\bar{\underline{Q}}_{ab}(t)}{\underline{Q}_{ba}(t)}\,\underline{\omega}_{ba,b}(t)\overbrace{\underset{\underline{Q}_{ab}(t)}{\bar{\underline{Q}}_{ba}(t)}\,\underline{Q}_{ba}(t)}^{1} = \frac{1}{2}\,\overbrace{\bar{\underline{Q}}_{ab}(t)\,\underline{\omega}_{ba,b}(t)\,\underline{Q}_{ab}(t)}^{\underline{\omega}_{ba,a}(t)}\,\underline{Q}_{ba}(t) = \frac{1}{2}\,\underline{\omega}_{ba,a}(t)\,\underline{Q}_{ba}(t)$$

$$\dot{\underline{Q}}_{ba}(t) = \frac{1}{2}\,\underline{\omega}_{ba,a}(t)\,\underline{Q}_{ba}(t)$$

Thus:

$$\dot{\underline{Q}}_{ba} = \frac{1}{2}\left(-\underline{w}_{ba,a}^{T}\,\underline{\varepsilon} + \eta\,\underline{w}_{ba,a} + \underline{w}_{ba,a}^{\times}\,\underline{\varepsilon}\right) \Rightarrow \begin{pmatrix} \dot{\underline{\varepsilon}} \\ \dot{\eta} \end{pmatrix} = \left[\begin{array}{c:c} \underline{w}_{ba,a}^{\times} & \underline{w}_{ba,a} \\ \hdashline -\underline{w}_{ba,a}^{T} & 0 \end{array}\right] \underline{Q}_{ba}(t)$$

Hence:

$$\dot{\underline{Q}}_{ba} = \frac{1}{2}\,\underline{\Omega}_{ba,a}\,\underline{Q}_{ba}$$

Notice that this is a linear equation!

where:

- $\underline{Q}_{ba} = \begin{pmatrix} \underline{\varepsilon} \\ \eta \end{pmatrix}$

- $\underline{\Omega}_{ba,a} \triangleq \left[\begin{array}{c:c} +\underline{w}_{ba,a}^{\times} & \underline{w}_{ba,a} \\ \hdashline -\underline{w}_{ba,a}^{T} & 0 \end{array}\right] = \left[\begin{array}{ccc:c} 0 & -w_3 & w_2 & w_1 \\ w_3 & 0 & -w_1 & w_2 \\ -w_2 & w_1 & 0 & w_3 \\ \hdashline -w_1 & -w_2 & -w_3 & 0 \end{array}\right]$, if $\underline{w}_{ba,a} = \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix}^{T}$